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Rigidity of manifolds with boundary under a lower weighted Ricci curvature bound

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Abstract

In the present thesis, we study comparison geometry of Riemannian manifolds with boundary under a lower weighted Ricci curvature bound. We develop the preceding studies for manifolds with boundary whose Ricci curvatures are bounded from below, and extend comparison geometric results obtained in such preceding studies to our weighted setting.

We mainly study manifolds with boundary under the setting in which the weighted Ricci curvature is bounded from below by a constant in the usual weighted case, and under the setting in which the weighted Ricci curvature is bounded from below by the weighting function. In each setting, we introduce an appropriate lower weighted mean curvature bound for the boundary. Under the curvature bounds, we prove various comparison geometric results, and conclude rigidity theorems. We obtain rigidity theorems for the inscribed radii, splitting theorems under the existence of a single ray, and volume growth rigidity theorems for the neighborhoods of the boundaries. Furthermore, we conclude rigidity theorems concerning the smallest Dirichlet eigenvalues of the weighted p-Laplacians on compact manifolds with boundary, and concerning the bottoms of the spectra of the weighted p-Laplacians. We also discuss segment inequalities and measure contraction properties on manifolds with boundary.

For the proof of our rigidity theorems, one of the key ingredients is to study Laplacian comparisons for the distance function from the boundary. We first show pointwise Laplacian comparisons under our curvature bounds. By using the pointwise Laplacian comparisons, we can prove inscribed radius rigidity theorems, splitting theorems, and volume growth rigidity theorems. We further develop the pointwise Laplacian comparisons, and establish global Laplacian comparisons in a distribution sense. By using the global Laplacian comparisons, we can prove eigenvalue rigidity theorems and spectrum rigidity theorems for the weighted p-Laplacians.

Most of the results stated in this thesis appear in the papers [45], [46], [47] and [48]. This thesis is written as a comprehensive paper including the contents in the papers [45], [46], [47] and [48].

Contents

Chapter 1

Introduction

In this thesis, we study comparison geometry of Riemannian manifolds with boundary under a lower weighted Ricci curvature bound. We prove several comparison geometric results under a lower weighted mean curvature bound for the boundary.

1.1 Background

For $n \geq 2$, let M be an n-dimensional Riemannian manifold with Riemannian metric g. We denote by Ric_g the Ricci curvature determined by g. Let $f : M \to \mathbb{R}$ be a smooth function. We denote by ∇f the gradient of f, and by $\operatorname{Hess} f$ the Hessian of f. For $N \in (-\infty, \infty]$, the *N-weighted Ricci curvature* Ric^N_f is defined as

$$\mathrm{Ric}^N_f := \mathrm{Ric}_g + \operatorname{Hess} f - \frac{\nabla f \otimes \nabla f}{N - n} \tag{1.1}$$

if $N \in (-\infty, \infty) \setminus \{n\}$; otherwise, if $N = \infty$, then $\mathrm{Ric}^N_f := \mathrm{Ric}_g + \operatorname{Hess} f$; if $N = n$, and if f is constant, then $\mathrm{Ric}^N_f := \mathrm{Ric}_g$; if $N = n$, and if f is not constant, then $\mathrm{Ric}^N_f := -\infty$ ([2]). The weighted Ricci curvature naturally appears in various fields, and it has been studied from each viewpoint; for instance, the geometry of Ricci solitons that play an important role in the theory of the Ricci flow, and the geometry of metric measure spaces originated from the works done by Lott and Villani [35], [36], and Sturm [49], [50]. In the study of the N-weighted Ricci curvature, the parameter N has been usually chosen from $[n, \infty]$. Recently, in the complemental case of $N \in (-\infty, n)$, several geometric properties have begun to be studied.

For Riemannian manifolds under a lower Ricci curvature bound, many comparison geometric results have already been obtained. We have known comparison results such as the Bonnet-Myers theorem and the Bishop-Gromov volume comparison theorem, and rigidity theorems such as the Cheng maximal diameter theorem and the Cheeger-Gromoll splitting theorem. Such comparison geometric results have been extended to Riemannian manifolds under a lower N-weighted Ricci curvature bound. In the usual weighted case of $N \in [n, \infty]$, for Riemannian manifolds whose N-weighted Ricci curvatures are bounded from below by constants, comparison geometric properties have been studied in [13], [33], [34], [43], [53], and so on. In the complemental case of $N \in (-\infty, n)$, Wylie [54], and Wylie and Yeroshkin [55] have studied comparison geometric properties. Wylie [54] has proved a splitting theorem of Cheeger-Gromoll type for manifolds of non-negative N-weighted Ricci curvature for $N \in (-\infty, 1]$. For $N \in (-\infty, \infty]$ and $\kappa \in \mathbb{R}$, Wylie and Yeroshkin [55] have introduced a curvature bound

$$\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}, \tag{1.2}$$

where (1.2) means that for every $x \in M$, and for every unit tangent vector u at x we have $\mathrm{Ric}^N_f(u) \geq (n-1)\kappa\, e^{\frac{-4f(x)}{n-1}}$. Under the curvature condition (1.2) for $N \in (-\infty, 1]$ and $\kappa \in \mathbb{R}$, Wylie and Yeroshkin [55] have obtained a maximal diameter theorem of Cheng type for a Riemannian metric $e^{\frac{-4f}{n-1}} g$, and a volume comparison theorem of Bishop-Gromov type for a weighted measure $e^{-\frac{n+1}{n-1}f}\,\mathrm{vol}_g$, where vol_g is the Riemannian volume measure determined by g.

For Riemannian manifolds with boundary under a lower Ricci curvature bound, for instance, Kasue [24], [25] has obtained several comparison geometric results, and conclude the following rigidity theorems: (1) an inscribed radius rigidity theorem; (2) a splitting theorem for non-compact manifolds with boundary; (3) an eigenvalue rigidity theorem for the Laplacians. It seems to be natural to consider whether such results can be extended to Riemannian manifolds with boundary under a lower weighted Ricci curvature bound.

1.2 Development

In [45], the author has developed the pioneering works done by Kasue [24], [25]. For manifolds with boundary under a lower Ricci curvature bound, the author has proved the following rigidity theorems: (1) a volume growth rigidity theorem for the neighborhoods of the boundaries; (2) a splitting theorem under the existence of a single ray; (3) a spectrum rigidity theorem for the p-Laplacians.

In [46], [47], [48], the author has established comparison geometry of manifolds with boundary under a lower weighted Ricci curvature bound, and extended comparison geometric results obtained in [24], [25], [45] to a weighted setting.

In [46], the author has studied manifolds with boundary whose N-weighted Ricci curvature are bounded from below by constants in the usual weighted case of $N \in [n, \infty]$. The author has proved the following rigidity theorems: (1) an inscribed radius rigidity theorem (see Theorem 1.1); (2) a splitting theorem (see Theorem 1.4); (3) a volume growth rigidity theorem (see Theorem 1.6); (4) an eigenvalue rigidity theorem for the weighted p-Laplacians (see Theorem 1.8); (5) a spectrum rigidity theorem for the weighted p-Laplacians (see Theorem 1.11).

In [47], the author has studied manifolds with boundary in the complementary case of $N \in (-\infty, 1]$ beyond the usual weighted case studied in [46], and obtained the following twisted rigidity theorems: (1) a twisted splitting theorem (see Theorem 1.5); (2) a twisted volume growth rigidity theorem (see Theorem 8.24); (3) a twisted eigenvalue rigidity theorem for the weighted p-Laplacians (see Theorem 9.9).

In [48], the author has developed comparison geometry under the curvature bound (1.2) for $N \in (-\infty, 1] \cup [n, \infty]$ and $\kappa \in \mathbb{R}$, and conclude the following twisted rigidity theorems: (1) twisted inscribed radius rigidity theorems (see Theorems 1.2 and 1.3); (2) a twisted splitting theorem (see Theorem 1.5); (3) twisted volume growth rigidity theorems (see Theorems 1.7 and 8.22); (4) twisted eigenvalue rigidity theorems for the weighted p-Laplacians (see Theorems 1.9 and 8.22); (5) a twisted spectrum rigidity theorem for the weighted p-Laplacians (see Theorem 1.11).

1.3 Setting

In this thesis, we study comparison geometry of Riemannian manifolds with boundary under a lower weighted Ricci curvature bound. For such manifolds with boundary, we introduce a reasonable curvature bound concerning a lower weighted mean curvature bound for the boundary, and conclude comparison geometric results. We will extend the results obtained in [24], [25], [45] to our weighted setting.

We now summarize our weighted setting in the present thesis as follows: For $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. Let $f : M \to \mathbb{R}$ be a smooth function. For $N \in (-\infty, \infty]$, the N-weighted Ricci curvature Ric_f^N is defined as (1.1). For a smooth function $K : M \to \mathbb{R}$, we mean by

$$\mathrm{Ric}_{f,M}^N \geq K$$

for every $x \in M$, and for every unit tangent vector u at x we have $\mathrm{Ric}_f^N(u) \geq K(x)$.

Let ∂M denote the boundary of M. For $z \in \partial M$, we denote by H_z the mean curvature on ∂M at z defined as the trace of the shape operator of the unit inner normal vector u_z at z. The f*-mean curvature* $H_{f,z}$ on ∂M at z is defined as

$$H_{f,z} := H_z + g\left((\nabla f)_z, u_z\right).$$

For a smooth function $\Lambda : \partial M \to \mathbb{R}$, we mean by

$$H_{f,\partial M} \geq \Lambda$$

for every $z \in \partial M$ we have $H_{f,z} \geq \Lambda(z)$. We consider the following two curvature bounds: for $N \in [n, \infty)$ and $\kappa, \lambda \in \mathbb{R}$, we have

$$\mathrm{Ric}_{f,M}^N \geq (N-1)\kappa, \quad H_{f,\partial M} \geq (N-1)\lambda; \tag{1.3}$$

for $N \in (-\infty, 1] \cup [n, \infty]$ and $\kappa, \lambda \in \mathbb{R}$, we have

$$\mathrm{Ric}_{f,M}^N \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}, \quad H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}. \tag{1.4}$$

We notice that for $N_1 \in (n, \infty]$ and $N_2 \in (-\infty, 1]$, if M has the curvature bound (1.4) for N_1, then it does for N_2 (see (1.1) and (1.4)).

We mainly study a Riemannian manifold M with boundary under the curvature bound (1.3) for $N \in [n, \infty)$ and $\kappa, \lambda \in \mathbb{R}$, and a Riemannian manifold M with boundary under the curvature bound (1.4) for $N \in (-\infty, 1] \cup [n, \infty]$ and $\kappa, \lambda \in \mathbb{R}$.

1.4 Main results

1.4.1 Inscribed radius rigidity

We denote by M_κ^n the simply connected n-dimensional space form with constant curvature κ. We say that κ and λ satisfy the *ball-condition* if there exists a closed geodesic ball $B_{\kappa,\lambda}^n$ in M_κ^n with non-empty boundary $\partial B_{\kappa,\lambda}^n$ such that $\partial B_{\kappa,\lambda}^n$ has a constant mean curvature $(n-1)\lambda$. We denote by $C_{\kappa,\lambda}$ the radius of $B_{\kappa,\lambda}^n$. Note that κ and λ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. Let $s_{\kappa,\lambda}(t)$ be a unique solution of the so-called

Jacobi-equation $\varphi''(t)+\kappa\varphi(t)=0$ with initial conditions $\varphi(0)=1$ and $\varphi'(0)=-\lambda$. We see that κ and λ satisfy the ball-condition if and only if the equation $s_{\kappa,\lambda}(t)=0$ has a positive solution; in particular, $C_{\kappa,\lambda}=\inf\{\,t>0 \mid s_{\kappa,\lambda}(t)=0\,\}$.

For the Riemannian distance d_M on M, the distance function $\rho_{\partial M}:M\to\mathbb{R}$ from ∂M is defined as $\rho_{\partial M}(x):=d_M(x,\partial M)$. The *inscribed radius $D(M,\partial M)$ of M* is defined by

$$D(M,\partial M):=\sup_{x\in M}\rho_{\partial M}(x).$$

One of our main results is the following inscribed radius rigidity theorem:

Theorem 1.1 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f:M\to\mathbb{R}$ be a smooth function. Let κ and λ satisfy the ball-condition. For $N\in[n,\infty)$, suppose $\mathrm{Ric}^N_{f,M}\geq(N-1)\kappa$ and $H_{f,\partial M}\geq(N-1)\lambda$. Then we have*

$$D(M,\partial M)\leq C_{\kappa,\lambda}. \tag{1.5}$$

If for some $x_0\in M$ we have $\rho_{\partial M}(x_0)=C_{\kappa,\lambda}$, then M is isometric to $B^n_{\kappa,\lambda}$, and $N=n$; in particular, f is constant on M.

Kasue [24] has proved Theorem 1.1 in the case where $f=0$ and $N=n$. We prove Theorem 1.1 in a similar way to that in [24].

Remark 1.1. M. Li [30] later than [24] has proved Theorem 1.1 in the case where $f=0$, $N=n$ and $\kappa=0$. H. Li and Wei have proved Theorem 1.1 in [29] when $\kappa=0$, and in [28] when $\kappa<0$. In [28] and [29], Theorem 1.1 in the specific cases have been proved in a similar way to that in [30].

Let g_f be a Riemannian metric on M defined by $g_f:=e^{\frac{-4f}{n-1}}g$. We denote by $d^{g_f}_M$ the Riemannian distance on M determined by g_f. We define a function $\rho^{g_f}_{\partial M}:M\to\mathbb{R}$ by $\rho^{g_f}_{\partial M}(x):=d^{g_f}_M(x,\partial M)$, and we put

$$D^{g_f}(M,\partial M):=\sup_{x\in M}\rho^{g_f}_{\partial M}(x). \tag{1.6}$$

We denote by $\mathrm{Int}\,M$ the interior of M. For $x\in\mathrm{Int}\,M$, let U_xM be the unit tangent sphere at x which can be identified with the $(n-1)$-dimensional standard unit sphere $\mathbb{S}^{n-1}$. For $u\in U_xM$, let $\gamma_u:[0,T)\to M$ be the geodesic with $\gamma_u(0)=x$ and $\gamma_u'(0)=u$. We define a function $\tau_x:U_xM\to(0,\infty]$ by

$$\tau_x(u):=\sup\{\,t>0 \mid \rho_x(\gamma_u(t))=t,\ \gamma_u([0,t))\subset\mathrm{Int}\,M\,\}, \tag{1.7}$$

where $\rho_x:M\to\mathbb{R}$ is the distance function from x defined as $\rho_x(y):=d_M(x,y)$. Let $s_{f,u}:[0,\tau_x(u)]\to[0,\infty]$ be a function defined by

$$s_{f,u}(t):=\int_0^t e^{\frac{-2f(\gamma_u(a))}{n-1}}\,da. \tag{1.8}$$

Let $s_\kappa(t)$ be a unique solution of the Jacobi-equation $\varphi''(t)+\kappa\varphi(t)=0$ with initial conditions $\varphi(0)=0$ and $\varphi'(0)=1$. We define a function $F_{\kappa,u}:[0,\tau_x(u)]\to\mathbb{R}$ by

$$F_{\kappa,u}(t):=\exp\left(\frac{f(\gamma_u(t))+f(x)}{n-1}\right)s_\kappa(s_{f,u}(t)). \tag{1.9}$$

For $l>0$, let $[0,l]\times_{F_\kappa}\mathbb{S}^{n-1}$ denote the twisted product Riemannian manifold $\left([0,l]\times\mathbb{S}^{n-1},dt^2+F^2_{\kappa,u}(t)ds^2_{n-1}\right)$, where ds^2_{n-1} is the standard metric over $\mathbb{S}^{n-1}$.

Under the curvature bound (1.4), we obtain the following:

Theorem 1.2 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Then we have*

$$D^{g_f}(M, \partial M) \leq C_{\kappa,\lambda}. \tag{1.10}$$

If for some $x_0 \in M$ we have $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$, then there exists $l > 0$ such that M is isometric to $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$; moreover, if $N \in (-\infty, 1) \cup [n, \infty]$, then f is constant on M; in particular, M is isometric to a closed geodesic ball in a simply connected space form.

For $\delta \in \mathbb{R}$, we define a function $\rho_{\partial M,\delta} : M \to \mathbb{R}$ by $\rho_{\partial M,\delta} := e^{-2\delta}\rho_{\partial M}$. We put

$$D_\delta(M, \partial M) := \sup_{x \in M} \rho_{\partial M,\delta}(x). \tag{1.11}$$

If f is bounded from above, then we conclude the following:

Theorem 1.3 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then we have*

$$D_\delta(M, \partial M) \leq C_{\kappa,\lambda}. \tag{1.12}$$

If for some $x_0 \in M$ we have $\rho_{\partial M,\delta}(x_0) = C_{\kappa,\lambda}$, then M is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on M.

1.4.2 Splitting theorems

For $z \in \partial M$, let $\gamma_z : [0, T) \to M$ be the geodesic with $\gamma_z(0) = z$ and $\gamma_z'(0) = u_z$. We define a function $\tau : \partial M \to (0, \infty]$ by

$$\tau(z) := \sup\{\, t > 0 \mid \rho_{\partial M}(\gamma_z(t)) = t \,\}. \tag{1.13}$$

For an interval I, and for a connected component ∂M_1 of the boundary ∂M, we denote by $I \times_{\kappa,\lambda} \partial M_1$ the warped product $\left(I \times \partial M_1, dt^2 + s^2_{\kappa,\lambda}(t)\, h\right)$, where h is the induced Riemannian metric over ∂M.

We obtain the following splitting theorem:

Theorem 1.4 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. If for some $z_0 \in \partial M$ we have $\tau(z_0) = \infty$, then M is isometric to $[0, \infty) \times_{\kappa,\lambda} \partial M$, and for all $z \in \partial M$ and $t \in [0, \infty)$ we have $(f \circ \gamma_z)(t) = f(z) + (N-n)\lambda t$.*

In the case where $f = 0$ and $N = n$, Kasue [24] has proved Theorem 1.4 under the assumption that M is non-compact and ∂M is compact (see also the work of Croke and Kleiner [12]). In that case, Theorem 1.4 itself has been proved by the author [45].

We define a function $\tau_f : \partial M \to (0, \infty]$ by

$$\tau_f(z) := \int_0^{\tau(z)} e^{\frac{-2f(\gamma_z(a))}{n-1}} \, da. \tag{1.14}$$

Furthermore, we define a function $s_{f,z} : [0, \tau(z)] \to [0, \tau_f(z)]$ by

$$s_{f,z}(t) := \int_0^{t} e^{\frac{-2f(\gamma_z(a))}{n-1}} \, da. \tag{1.15}$$

Let $F_{\kappa,\lambda,z} : [0, \tau(z)] \to \mathbb{R}$ denote a function defined by

$$F_{\kappa,\lambda,z}(t) := \exp\left(\frac{f(\gamma_z(t)) - f(z)}{n-1} \right) s_{\kappa,\lambda}(s_{f,z}(t)). \tag{1.16}$$

For an interval I, and for a connected component ∂M_1 of the boundary ∂M, we denote by $I \times_{F_{\kappa,\lambda}} \partial M_1$ the twisted product $\big(I \times \partial M_1, dt^2 + F^2_{\kappa,\lambda,z}(t)\, h\big)$.

Under the curvature bound (1.4), we provide the following:

Theorem 1.5 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function bounded from above. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose that we have $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. If for some point $z_0 \in \partial M$ we have $\tau(z_0) = \infty$, then M is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if we have $N \in (-\infty, 1) \cup [n, \infty]$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant.*

In the case where $\kappa = 0$ and $\lambda = 0$, Theorem 1.5 has been obtained by the author [47] when $N \in (-\infty, 1]$.

In Theorem 1.5, by applying a splitting theorem of Cheeger-Gromoll type ([11]) obtained by Wylie [54] to the boundary, we conclude a multi-splitting theorem (see Section 7.4). We generalize splitting theorems for manifolds with boundary whose boundaries are disconnected that have been studied by Kasue [24] (and Croke and Kleiner [12], Ichida [22]) (see Section 7.5).

1.4.3 Volume growth rigidity

For $\kappa, \lambda \in \mathbb{R}$, we put $\bar{C}_{\kappa,\lambda} := C_{\kappa,\lambda}$ if κ and λ satisfy the ball-condition; otherwise, $\bar{C}_{\kappa,\lambda} := \infty$. Let $I_{\kappa,\lambda} := [0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$. For $N \in (1, \infty)$, we define functions $\bar{s}_{\kappa,\lambda}$, $s_{N,\kappa,\lambda} : [0, \infty) \to \mathbb{R}$ by

$$\bar{s}_{\kappa,\lambda}(t) := \begin{cases} s_{\kappa,\lambda}(t) & \text{if } t < \bar{C}_{\kappa,\lambda}, \\ 0 & \text{if } t \geq \bar{C}_{\kappa,\lambda}, \end{cases} \quad s_{N,\kappa,\lambda}(r) := \int_0^r \bar{s}^{N-1}_{\kappa,\lambda}(a)\, da. \tag{1.17}$$

For a smooth function $\phi : M \to \mathbb{R}$, we put $m_\phi := e^{-\phi}\, \mathrm{vol}_g$. For the Riemannian volume measure vol_h on ∂M induced from h, let $m_{f,\partial M} := e^{-f|_{\partial M}}\, \mathrm{vol}_h$. For $r > 0$, we put $B_r(\partial M) := \{x \in M \mid \rho_{\partial M}(x) \leq r\}$.

For the neighborhoods of the boundaries, we prove absolute volume comparison results of Heintze-Karcher type ([20]), and relative volume comparison results of Bishop-Gromov type ([17], [18]) (see Sections 8.3 and 8.4). We prove them by using comparison results for volume elements, and a geometric study of the cut locus for the boundary. Furthermore, we obtain volume growth rigidity theorems concerning the equality cases of the volume comparison results (see Section 8.5).

One of the volume growth rigidity theorems is the following:

Theorem 1.6 ([46])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. For $N \in [n, \infty)$, suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. If we have*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} \geq m_{f,\partial M}(\partial M), \tag{1.18}$$

then M is isometric to the warped product $I_{\kappa,\lambda} \times_{\kappa,\lambda} \partial M$, and for every point $z \in \partial M$ we have $f \circ \gamma_z = f(z) - (N-n)\log s_{\kappa,\lambda}$ on $I_{\kappa,\lambda}$; moreover, if κ and λ satisfy the ball-condition, then M is isometric to $B^n_{\kappa,\lambda}$ and $N = n$; in particular, f is constant.

The author [45] has proved Theorem 1.6 in the case where $f = 0$ and $N = n$.

Remark 1.2. Under the same setting as in Theorem 1.6, we always have the following volume comparison inequality (see Lemma 8.11):

$$\limsup_{r\to\infty} \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} \leq m_{f,\partial M}(\partial M). \tag{1.19}$$

Theorem 1.6 is concerned with rigidity phenomena.

For $x \in M$, we say that $z \in \partial M$ is a *foot point on ∂M of x* if $d_M(x, z) = \rho_{\partial M}(x)$. Notice that every point in M has at least one foot point on ∂M. We define a function $\rho_{\partial M,f} : M \to \mathbb{R}$ by

$$\rho_{\partial M,f}(x) := \inf_{z\in\partial M} \int_0^{\rho_{\partial M}(x)} e^{\frac{-2f(\gamma_z(a))}{n-1}}\, da, \tag{1.20}$$

where the infimum is taken over all foot points z on ∂M of x. For $r > 0$, we put $B_{r,f}(\partial M) := \{\, x \in M \mid \rho_{\partial M,f}(x) \leq r \,\}$.

Under the curvature bound (1.4), we obtain the following rigidity theorem:

Theorem 1.7 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let us suppose that κ and λ do not satisfy the ball-condition. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. If we have*

$$\liminf_{r\to\infty} \frac{m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M))}{s_{n,\kappa,\lambda}(r)} \geq m_{f,\partial M}(\partial M), \tag{1.21}$$

then M is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1) \cup [n, \infty]$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant.

Remark 1.3. Under the same setting as in Theorem 1.7, we always have the following inequality (see Lemma 8.12):

$$\limsup_{r\to\infty} \frac{m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M))}{s_{n,\kappa,\lambda}(r)} \leq m_{f,\partial M}(\partial M). \tag{1.22}$$

Theorem 1.7 is concerned with rigidity phenomena.

Remark 1.4. If κ and λ satisfy the ball-condition, then the author does not know whether a similar result to Theorem 1.7 holds (see Remark 8.7).

Under the curvature bound (1.4), we also conclude a volume growth rigidity theorem in the case where f is bounded from above (see Theorem 8.22). We prove a volume growth rigidity theorem in the case where the N-weighted Ricci curvature is bounded from below by a constant for $N \in (-\infty, 1]$ (see Theorem 8.24).

1.4.4 Eigenvalue rigidity

Let $p \in [1, \infty)$, and let $\phi : M \to \mathbb{R}$ be a smooth function. The $(1, p)$-*Sobolev space* $W_0^{1,p}(M, m_\phi)$ *on* (M, m_ϕ) *with compact support* is defined as the completion of the set of all smooth functions on M whose support is compact and contained in $\operatorname{Int} M$ with respect to the standard $(1, p)$-Sobolev norm. We denote by $\| \cdot \|$ the standard norm induced from g, and by div the divergence with respect to g. The (ϕ, p)-*Laplacian* $\Delta_{\phi,p}\, \psi$ for $\psi \in W_0^{1,p}(M, m_\phi)$ is defined by

$$\Delta_{\phi,p}\, \psi := -e^{\phi} \operatorname{div} \left(e^{-\phi} \| \nabla \psi \|^{p-2}\, \nabla \psi \right)$$

as a distribution on $W_0^{1,p}(M, m_\phi)$. A real number μ is said to be a (ϕ, p)-*Dirichlet eigenvalue* on M if there exists a non-zero $\psi \in W_0^{1,p}(M, m_\phi)$ such that the equality $\Delta_{\phi,p} \psi = \mu |\psi|^{p-2}\, \psi$ holds on $\operatorname{Int} M$ in a distribution sense on $W_0^{1,p}(M, m_\phi)$. The *Rayleigh quotient* $R_{\phi,p}(\psi)$ for $\psi \in W_0^{1,p}(M, m_\phi) \setminus \{0\}$ is defined as

$$R_{\phi,p}(\psi) := \frac{\int_M \, \| \nabla \psi \|^p \, d\, m_\phi}{\int_M \, |\psi|^p \, d\, m_\phi}.$$

We put $\mu_{\phi,p}(M) := \inf_\psi R_{\phi,p}(\psi)$, where the infimum is taken over all non-zero functions in $W_0^{1,p}(M, m_\phi)$. The value $\mu_{\phi,2}(M)$ is equal to the infimum of the spectrum of $\Delta_{\phi,2}$ on (M, m_ϕ). If M is compact, and if $p \in (1, \infty)$, then $\mu_{\phi,p}(M)$ is equal to the infimum of the set of all (ϕ, p)-Dirichlet eigenvalues on M. If ϕ is a constant function on M, then $\mu_{\phi,p}(M)$ can be written by $\mu_{0,p}(M)$.

Let $p \in (1, \infty)$. For $N \in (1, \infty)$ and $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, let $\mu_{p,N,\kappa,\lambda,D}$ denote the positive minimum real number μ such that there exists a non-zero function $\varphi : [0, D] \to \mathbb{R}$ such that

$$\left(|\varphi'(t)|^{p-2} \varphi'(t) \right)' + (N - 1) \frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)} \left(|\varphi'(t)|^{p-2} \varphi'(t) \right) + \mu\, |\varphi(t)|^{p-2} \varphi(t) = 0, \tag{1.23}$$

$$\varphi(0) = 0, \; \varphi'(D) = 0.$$

We recall the notion of model spaces that has been introduced by Kasue in [25]. We say that κ and λ satisfy the *model-condition* if the equation $s'_{\kappa,\lambda}(t) = 0$ has a positive solution. We see that κ and λ satisfy the model-condition if and only if either (1) $\kappa > 0$ and $\lambda < 0$; (2) $\kappa = 0$ and $\lambda = 0$; or (3) $\kappa < 0$ and $\lambda \in (0, \sqrt{|\kappa|})$.

Let κ and λ satisfy the ball-condition or the model-condition. Suppose that M is compact. If κ and λ satisfy the model-condition, then we define a positive number $D_{\kappa,\lambda}(M)$ as follows: If $\kappa = 0$ and $\lambda = 0$, then $D_{\kappa,\lambda}(M) := D(M, \partial M)$; otherwise, $D_{\kappa,\lambda}(M) := \{ t > 0 \mid s'_{\kappa,\lambda}(t) = 0 \}$. We say that M is isometric to a (κ, λ)-*equational model space* if M is isometric to either (1) for κ and λ satisfying the ball-condition, the closed geodesic ball $B^n_{\kappa,\lambda}$; (2) for κ and λ satisfying the model-condition, and for a connected component ∂M_1 of ∂M, the warped product $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M_1$; or (3) for κ and λ satisfying the model-condition, and for an involutive isometry σ of ∂M without fixed points, the quotient space $\left([0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M \right) / G_\sigma$, where G_σ is the isometry group on $[0, 2D_{\kappa,\lambda}(M)] \times_{\kappa,\lambda} \partial M$ whose elements consist of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, z) := (2D_{\kappa,\lambda}(M) - t, \sigma(z))$.

We conclude the following eigenvalue rigidity theorem:

Theorem 1.8 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1,\infty)$. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D(M,\partial M) \leq D$. Then*

$$\mu_{f,p}(M) \geq \mu_{p,N,\kappa,\lambda,D}. \tag{1.24}$$

If the equality in (1.24) *holds, then M is isometric to a (κ,λ)-equational model space; more precisely, the following hold:*

(1) *if $D = \bar{C}_{\kappa,\lambda}$, then κ and λ satisfy the ball-condition, M is isometric to $B^n_{\kappa,\lambda}$, and $N = n$; in particular, f is constant on M;*

(2) *if $D \in (0, \bar{C}_{\kappa,\lambda})$, then κ and λ satisfy the model-condition, M is isometric to a (κ,λ)-equational model space, and $f \circ \gamma_z = f(z) - (N-n)\log s_{\kappa,\lambda}$ on $[0, D_{\kappa,\lambda}(M)]$ for all $z \in \partial M$.*

Kasue [25] has proved Theorem 1.8 in the case where $f = 0$, $N = n$ and $p = 2$. It seems that the method of the proof in [25] does not work in our non-linear case of $p \neq 2$ (see Remark 9.6). We prove Theorem 1.8 by using a global Laplacian comparison result for the distance function from the boundary (see Proposition 5.2), and an inequality of Picone type for the p-Laplacian (see Lemma 9.1).

We say that the smooth function f is *∂M-radial* if there exists a smooth function $\phi_f : [0,\infty) \to \mathbb{R}$ such that $f = \phi_f \circ \rho_{\partial M}$ on M. For the function $\rho_{\partial M,f} : M \to \mathbb{R}$ defined as (1.20), we put

$$D_f(M,\partial M) := \sup_{x \in M} \rho_{\partial M,f}(x). \tag{1.25}$$

Under the curvature bound (1.4), we establish the following rigidity theorem:

Theorem 1.9 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact, and f is ∂M-radial. Let $p \in (1,\infty)$. For $N \in (-\infty, 1] \cup [n,\infty]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D_f(M,\partial M) \leq D$. Then we have*

$$\mu_{\frac{n+1}{n-1}f,p}(M) \geq e^{-2p\delta}\, \mu_{p,n,\kappa,\lambda,D}. \tag{1.26}$$

If the equality in (1.26) *holds, then M is isometric to a $\left(\kappa e^{-4\delta}, \lambda e^{-2\delta}\right)$-equational model space, and $f = (n-1)\delta$ on M.*

Remark 1.5. In Theorem 1.9, the author does not know whether the assumption that f is ∂M-radial can be dropped.

For real numbers $\kappa, \lambda \in \mathbb{R}$, we say that κ and λ satisfy the *convex-ball-condition* if they satisfy the ball-condition and λ is non-negative. If κ is non-positive, then we see that κ and λ satisfy the convex-ball-condition if and only if they satisfy the ball-condition.

In the case where f is not necessarily ∂M-radial, we obtain the following:

Theorem 1.10 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1, \infty)$. Let κ and λ satisfy the convex-ball-condition. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then we have*

$$\mu_{f,p}(M) \geq e^{-2p\delta} \mu_{0,p}(B^n_{\kappa,\lambda}). \tag{1.27}$$

If the equality in the inequality (1.27) *holds, then M is isometric to $B^n_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on M.*

Remark 1.6. The author does not know whether Theorem 1.10 holds under the assumption that κ and λ satisfy the ball-condition.

We also provide an eigenvalue rigidity theorem under the assumption that the N-weighted Ricci curvature is bounded from below by a constant for $N \in (-\infty, 1]$ (see Theorem 9.9).

1.4.5 Spectrum rigidity

We show volume estimates for a relatively compact domain in M (see Section 10.1). From the volume estimates we deduce a lower bound for $\mu_{f,p}$ (see Section 10.2).

By combining the estimate for $\mu_{f,p}$ and Theorem 1.4, we conclude the following rigidity theorem for manifolds with boundary that are not necessarily compact:

Theorem 1.11 ([46])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary. Let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then we have*

$$\mu_{f,p}(M) \geq \left(\frac{(N-1)\lambda}{p} \right)^p. \tag{1.28}$$

If the equality in the inequality (1.28) *holds, then M is isometric to $[0, \infty) \times_{\kappa,\lambda} \partial M$, and for all $z \in \partial M$ and $t \in [0, \infty)$ we have $(f \circ \gamma_z)(t) = f(z) + (N-n)\lambda t$.*

The author [45] has obtained Theorem 1.11 in the case where $f = 0$ and $N = n$.

Under the curvature bound (1.4), from Theorem 1.5 we derive the following:

Theorem 1.12 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary. Let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1] \cup [n, \infty]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then we have*

$$\mu_{f,p}(M) \geq e^{-2p\delta} \left(\frac{(n-1)\lambda}{p} \right)^p. \tag{1.29}$$

If the equality in the inequality (1.29) *holds, then M is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1) \cup [n, \infty]$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0, \infty)$.*

1.5 Plan

By the monotonicity of the curvature bound (1.4) with respect to N, for the proofs of our main theorems under the curvature bound (1.4), it suffices to consider the case of $N \in (-\infty, 1]$ (see Subsection 1.3). In what follows, when we prove comparison geometric results under the curvature bound (1.4) for $N \in (-\infty, 1] \cup [n, \infty]$, we study a Riemannian manifold M with boundary with the curvature bound (1.4) for $N \in (-\infty, 1]$. All the claims for $N \in (-\infty, 1]$ discussed below are also valid in the usual weighted case of $N \in [n, \infty]$.

In the proof of our main theorems, the key tools are Laplacian comparisons for the distance function $\rho_{\partial M}$ from the boundary. From pointwise Laplacian comparisons for the distance function $\rho_{\partial M}$ we derive inscribed radius rigidity theorems, splitting theorems and volume growth rigidity theorems. In order to conclude rigidity theorems for the weighted p-Laplacians, we develop the pointwise Laplacian comparisons, and establish global Laplacian comparisons in a distribution sense. The key points are to find a suitable assumption on the function f, and to choose an appropriate weighted p-Laplace operator and a weighted distance function. In the case where M has the curvature bound (1.3), we can obtain a global Laplacian comparison result for the operator $\Delta_{f,p}$ and the function $\rho_{\partial M}$ without any additional assumption on the function f. In the case where M has the curvature bound (1.4), by assuming that $f \leq (n-1)\delta$ for $\delta \in \mathbb{R}$, we can conclude a global Laplacian comparison result for the operator $\Delta_{f,p}$ and the function $\rho_{\partial M,\delta}$. Furthermore, in that case, assuming that f is ∂M-radial enables us to prove such a comparison result for the operator $\Delta_{\frac{n+1-2p}{n-1},p}$ and the function $\rho_{\partial M,f}$.

1.6 Organization

In Chapter 2, we prepare some notations and recall the basic facts for weighted Riemannian manifolds with boundary.

In Chapter 3, we study properties of the cut locus for the boundary. It seems that such properties are well-known. For the sake of the readers, we give proofs.

In Chapters 4 and 5, we study Laplacian comparisons for the distance function from the boundary. In Chapter 4, we show pointwise Laplacian comparison results (see Section 4.2), and rigidity results in the equality cases (see Section 4.4). In Chapter 5, we develop the pointwise Laplacian comparisons, and establish global Laplacian comparisons in a distribution sense.

In Chapter 6, we prove Theorems 1.1, 1.2 and 1.3. In Chapter 7, we prove Theorems 1.4 and 1.5, and study the variants of the splitting theorems. In Chapter 8, we prove various volume comparison theorems around the boundaries, and conclude Theorems 1.6 and 1.7. In Chapter 9, we prove Theorems 1.8, 1.9 and 1.10, and study explicit lower bounds of the smallest Dirichlet eigenvalues. In Chapter 10, we prove Theorems 1.11 and 1.12.

In Chapter 11, we study segment inequalities of Cheeger-Colding type on manifolds with boundary. Cheeger and Colding [10] have proved a segment inequality for complete Riemannian manifolds under a lower Ricci curvature bound. They have mentioned in [10] that their segment inequality gives a lower bound for the smallest Dirichlet eigenvalue for the Laplacian on a closed ball. From our segment inequality we derive a lower bound for $\mu_{0,p}$ (see Section 11.3).

In Chapter 12, we study measure contraction properties of manifolds with boundary. For metric measure spaces, Ohta [40] and Sturm [50] have independently introduced the so-called measure contraction property that is equivalent to a lower Ricci curvature bound for manifolds without boundary. We prove a measure contraction inequality around the boundary (see Proposition 12.4). By using our measure contraction inequality, we give another proof of a relative volume comparison theorem.

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Chapter 2

Preliminaries

In the present chapter, we review basics of Riemannian manifolds with boundary, and present some basic facts for weighted Riemannian manifolds with boundary. We refer to [44] for the basics of Riemannian manifolds with boundary.

2.1 Riemannian manifolds with boundary

Let M be a connected Riemannian manifold with boundary with Riemannian metric g. Let d_M be the Riemannian distance on M induced from g. For $r > 0$ and $A \subset M$, we denote by $U_r(A)$ the open r-neighborhood of A in M, and by $B_r(A)$ the closed one. For an interval I, we say that a curve $\gamma : I \to M$ is a *minimal geodesic* if for all $t_1, t_2 \in I$ we have $d_M(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. If the metric space (M, d_M) is complete, then the Hopf-Rinow theorem for length spaces (see e.g., Theorem 2.5.23 in [5]) tells us that (M, d_M) is a proper, geodesic space; namely, all closed bounded subsets of M are compact, and for every pair of points in M we have a minimal geodesic connecting them.

For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. For each i, the boundary ∂M_i carries the induced Riemannian metric h_i. We say that a homeomorphism $\Phi : M_1 \to M_2$ is a *Riemannian isometry with boundary* from M_1 to M_2 if Φ satisfies the following conditions:

(1) $\Phi|_{\mathrm{Int}\, M_1} : \mathrm{Int}\, M_1 \to \mathrm{Int}\, M_2$ is smooth, and $(\Phi|_{\mathrm{Int}\, M_1})^*(g_2) = g_1$;

(2) $\Phi|_{\partial M_1} : \partial M_1 \to \partial M_2$ is smooth, and $(\Phi|_{\partial M_1})^*(h_2) = h_1$.

If $\Phi : M_1 \to M_2$ is a Riemannian isometry with boundary, then the inverse Φ^{-1} is also a Riemannian isometry with boundary.

For manifolds without boundary, we have already known the following (see e.g., Theorem 11.1 in [21]):

Lemma 2.1. *For $i = 1, 2$, let M_i be connected Riemannian manifolds (without boundary) with Riemannian metric g_i, and let d_{M_i} be the Riemannian distances on M_i. Suppose that a map $\Phi : M_1 \to M_2$ is an isometry between the metric spaces (M_1, d_{M_1}) and (M_2, d_{M_2}). Then Φ is smooth, and $\Phi^* g_{M_2} = g_{M_1}$. Namely, Φ is a Riemannian isometry from (M_1, g_{M_1}) to (M_2, g_{M_2}).*

For manifolds with boundary, we have the following (see e.g., [45]):

Lemma 2.2. *For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. Then there exists a Riemannian isometry with boundary from M_1 to M_2 if and only if the metric space (M_1, d_{M_1}) is isometric to (M_2, d_{M_2}).*

Proof. For $i = 1, 2$, we denote by $\|\cdot\|_{g_i}$ and by $\|\cdot\|_{h_i}$ the canonical norms determined by g_i and by h_i on M_i and on ∂M_i, respectively. For a piecewise smooth curve γ in M_i, let $L_{g_i}(\gamma)$ denote the length of γ determined by g_i.

First, we show that if $\Phi : M_1 \to M_2$ is a Riemannian isometry with boundary, then it is an isometry between the metric spaces (M_1, d_{M_1}) and (M_2, d_{M_2}). It suffices to show that Φ is a 1-Lipschitz map from (M_1, d_{M_1}) to (M_2, d_{M_2}). Take $x, y \in M_1$. For $\epsilon > 0$, take a piecewise smooth curve $\gamma : [0, l] \to M_1$ with $L_{g_1}(\gamma) < d_{M_1}(x, y) + \epsilon$. For $t \in [0, l]$ at which γ is smooth, if $\gamma(t) \in \operatorname{Int} M_1$, then $\|(\Phi \circ \gamma)'(t)\|_{g_2} = \|\gamma'(t)\|_{g_1}$, and if $\gamma(t) \in \partial M_1$, then $\|(\Phi \circ \gamma)'(t)\|_{h_2} = \|\gamma'(t)\|_{h_1}$. It follows that $L_{g_2}(\Phi \circ \gamma)$ is equal to $L_{g_1}(\gamma)$. We have $d_{M_2}(\Phi(x), \Phi(y)) < d_{M_1}(x, y) + \epsilon$. This proves that Φ is a 1-Lipschitz map.

Next, we show that if $\Psi : M_1 \to M_2$ is an isometry between the metric spaces (M_1, d_{M_1}) and (M_2, d_{M_2}), then it is a Riemannian isometry with boundary. To do this, we first show that $\Psi|_{\operatorname{Int} M_1}$ is smooth on $\operatorname{Int} M_1$, and $(\Psi|_{\operatorname{Int} M_1})^*(g_2) = g_1$. For a fixed $x \in \operatorname{Int} M_1$, take a sufficiently small $r > 0$ such that $U_r(x)$ and $U_r(\Psi(x))$ are strongly convex in $(\operatorname{Int} M_1, g_1)$ and in $(\operatorname{Int} M_2, g_2)$, respectively. In this case, $\Psi|_{U_r(x)}$ is an isometry between the metric subspaces $U_r(x)$ and $U_r(\Psi(x))$. By applying Lemma 2.1 to the open Riemannian submanifolds $U_r(x)$ and $U_r(\Psi(x))$, $\Psi|_{U_r(x)}$ is a smooth Riemannian isometry. Hence $\Psi|_{\operatorname{Int} M_1}$ is smooth, and $(\Psi|_{\operatorname{Int} M_1})^*(g_2) = g_1$.

We second show that $\Psi|_{\partial M_1}$ is smooth on ∂M_1, and $(\Psi|_{\partial M_1})^*(h_2) = h_1$. To do this, we prove that $\Psi|_{\partial M_1}$ is an isometry between the metric spaces $(\partial M_1, d_{\partial M_1})$ and $(\partial M_2, d_{\partial M_2})$, where $d_{\partial M_1}$ and $d_{\partial M_2}$ are the Riemannian distances on ∂M_1 and on ∂M_2, respectively. It suffices to show that $\Psi|_{\partial M_1}$ is a 1-Lipschitz map from $(\partial M_1, d_{\partial M_1})$ to $(\partial M_2, d_{\partial M_2})$. Pick $\hat{z}, \check{z} \in \partial M_1$. For $\epsilon > 0$, take a piecewise smooth curve $\gamma : [0, l] \to \partial M_1$ satisfying $L_{h_1}(\gamma) < d_{\partial M_1}(\hat{z}, \check{z}) + \epsilon$. Fix $t \in [0, l]$ at which γ is smooth. From the fact that Ψ is an isometry between (M_1, d_{M_1}) and (M_2, d_{M_2}) we deduce

$$\begin{aligned}\|\gamma'(t)\|_{h_1} = \|\gamma'(t)\|_{g_1} &= \lim_{\alpha \to 0} \frac{d_{M_1}(\gamma(t), \gamma(t + \alpha))}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{d_{M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t + \alpha))}{\alpha}.\end{aligned} \tag{2.1}$$

Since ∂M_2 is smooth, and since h_2 is induced from g_2, for each $z_0 \in \partial M_2$ we see

$$\lim_{z \to z_0} \frac{d_{\partial M_2}(z_0, z)}{d_{M_2}(z_0, z)} = 1,$$

where the limit is taken with respect to $d_{\partial M_2}$. This implies

$$\lim_{\alpha \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t + \alpha))}{d_{M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t + \alpha))} = 1. \tag{2.2}$$

Combining (2.1) and (2.2) leads us to

$$\|\gamma'(t)\|_{h_1} = \lim_{\alpha \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t + \alpha))}{\alpha}. \tag{2.3}$$

Integrating the both sides of (2.3) on $[0, l]$ with respect to t, we have

$$L_{h_1}(\gamma) = \int_0^l \lim_{\alpha \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\alpha))}{\alpha} \, dt. \tag{2.4}$$

The right hand side of (2.4) is equal to the length of $\Psi \circ \gamma$ with respect to $d_{\partial M_2}$ (see e.g., Section 2.7 in [5]), and is greater than or equal to $d_{\partial M_2}(\Psi(\hat{z}), \Psi(\check{z}))$. Hence $d_{\partial M_2}(\Psi(\hat{z}), \Psi(\check{z})) < d_{\partial M_1}(\hat{z}, \check{z}) + \epsilon$; in particular, $\Psi|_{\partial M_1}$ is a 1-Lipschitz map. We conclude that $\Psi|_{\partial M_1}$ is an isometry between $(\partial M_1, d_{\partial M_1})$ and $(\partial M_2, d_{\partial M_2})$. By applying Lemma 2.1 to ∂M_1 and ∂M_2, $\Psi|_{\partial M_1}$ is smooth, and $(\Psi|_{\partial M_1})^*(h_2) = h_1$.

We complete the proof of Lemma 2.2. □

2.2 Jacobi fields orthogonal to the boundary

Let M be a connected Riemannian manifold with boundary with Riemannian metric g. For $x \in \operatorname{Int} M$, let $T_x M$ and $U_x M$ be the tangent space at x and the unit tangent sphere at x, respectively. For $z \in \partial M$, and for the tangent space $T_z \partial M$ at z on ∂M, let $T_z^\perp \partial M$ be the orthogonal complement of $T_z \partial M$ in the tangent space at z on M. For $u \in T_z^\perp \partial M$, and for the second fundamental form S of ∂M, let $A_u : T_z \partial M \to T_z \partial M$ denote the *shape operator* for u defined by

$$g(A_u v, w) := g(S(v, w), u).$$

For the unit inner normal vector u_z at z, the *mean curvature* H_z at z is defined as $H_z := \operatorname{trace} A_{u_z}$. We denote by $\gamma_z : [0, T) \to M$ the geodesic with $\gamma_z(0) = z$ and $\gamma_z'(0) = u_z$. We say that a Jacobi field Y along γ_z is a *∂M-Jacobi field* if Y satisfies the following initial conditions:

$$Y(0) \in T_z \partial M, \quad Y'(0) + A_{u_z} Y(0) \in T_z^\perp \partial M.$$

We say that $\gamma_z(t_0)$ is a *conjugate point* of ∂M along γ_z if there exists a non-zero ∂M-Jacobi field Y along γ_z with $Y(t_0) = 0$. We denote by $\tau_1(z)$ the first conjugate value for ∂M along γ_z. It is well-known that for all $t > \tau_1(z)$ we have $t > \rho_{\partial M}(\gamma_z(t))$, and that the function $T_1 : \partial M \to (0, \infty]$ defined by $T_1(z) := \tau_1(z)$ is continuous.

For the normal tangent bundle $T^\perp \partial M := \bigcup_{z \in \partial M} T_z^\perp \partial M$ of ∂M, let $0(T^\perp \partial M)$ be the zero-section $\bigcup_{z \in \partial M} \{ 0_z \in T_z^\perp \partial M \}$ of $T^\perp \partial M$. On an open neighborhood of $0(T^\perp \partial M)$ in $T^\perp \partial M$, the normal exponential map $\exp^\perp$ of ∂M is defined as $\exp^\perp(z, u) := \gamma_z(\|u\|)$ for $z \in \partial M$ and $u \in T_z^\perp \partial M$.

For the differential $D \exp^\perp$ of $\exp^\perp$, we recall the following Gauss lemma:

Lemma 2.3. *Take $z \in \partial M$. For $t \in [0, \tau_1(z))$, take $u \in T_{(z, t u_z)} T^\perp \partial M$. Then*

$$g\left(D \exp^\perp_{(z, t u_z)} u, \gamma_z'(t)\right) = g(u^\perp, u_z),$$

where for the direct sum decomposition $T_{(z, t u_z)} T^\perp \partial M = T_z \partial M \oplus T_z^\perp \partial M$, we denote by $u^\perp$ the $T_z^\perp \partial M$-component of u.

2.3 Weighted Riemannian manifolds with boundary

Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ be a smooth function. For a smooth function $\psi : M \to \mathbb{R}$, the *weighted Laplacian* $\Delta_f \psi$ *for* ψ is defined by

$$\Delta_f \psi := \Delta \psi + g(\nabla f, \nabla \psi),$$

where $\Delta \psi$ denotes the Laplacian for ψ defined as the minus of the trace of its Hessian. We see that Δ_f coincides with the $(f, 2)$-Laplacian $\Delta_{f,2}$.

The following formula of Bochner type seems to be well-known (see e.g., [13], and Chapter 14 in [52]).

Proposition 2.4 ([13]). *For every smooth function* $\varphi : M \to \mathbb{R}$*, we have*

$$-\frac{1}{2} \Delta_f \|\nabla \psi\|^2 = \mathrm{Ric}_f^\infty(\nabla \psi) + \|\operatorname{Hess} \psi\|^2 - g\left(\nabla \Delta_f \psi, \nabla \psi\right),$$

where $\|\operatorname{Hess}\psi\|$ *denotes the Hilbert-Schmidt norm of* $\operatorname{Hess}\psi$*.*

Let $\psi : M \to \mathbb{R}$ be a continuous function, and let U be a domain contained in $\operatorname{Int} M$. For $x \in U$, and for a function $\hat{\psi}$ defined on an open neighborhood of x, we say that $\hat{\psi}$ is a *support function of* ψ *at* x if we have $\hat{\psi}(x) = \psi(x)$ and $\hat{\psi} \leq \psi$. We also say that ψ is f*-subharmonic on* U if for every $x \in U$, and for every $\epsilon > 0$, there exists a smooth, support function $\psi_{x,\epsilon}$ of ψ at x such that $\Delta_f \psi_{x,\epsilon}(x) \leq \epsilon$.

For f-subharmonic functions, we recall the following maximal principle of Calabi type (see e.g., [6], and Lemma 2.4 in [13]).

Lemma 2.5 ([6]). *For a domain contained* U *in* $\operatorname{Int} M$*, if an* f*-subharmonic function on* U *takes the maximal value at a point in* U*, then it must be constant.*

2.4 Laplacian comparisons from a single point

Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. For $N \in (1, \infty)$, and for the diameter C_κ of M_κ^n, we define a function $H_{N,\kappa} : (0, C_\kappa) \to \mathbb{R}$ by

$$H_{N,\kappa}(t) := -(N-1) \frac{s_\kappa'(t)}{s_\kappa(t)}. \tag{2.5}$$

For the distance function from a single point, Qian [43] has obtained a Laplacian comparison inequality under a lower N-weighted Ricci curvature bound in the case of $N \in [n, \infty)$ (see equation 7 in [43]). In our setting, the Laplacian comparison inequality holds in the following form:

Lemma 2.6 ([43]). *Take* $x \in \operatorname{Int} M$ *and* $u \in U_x M$*. For* $N \in [n, \infty)$*, suppose* $\mathrm{Ric}_{f,M}^N \geq (N-1)\kappa$*. Then for all* $t \in (0, \tau_x(u))$ *we have*

$$\Delta_f \rho_x(\gamma_u(t)) \geq H_{N,\kappa}(t), \tag{2.6}$$

where τ_x *denotes the function defined as* (1.7)*.*

Remark 2.1. Assume that for some $t_0 \in (0, \tau_x(u))$ the equality in (2.6) holds. Choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of T_xM with $e_{u,n} = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0_x$ and $Y'_{u,i}(0) = e_{u,i}$. Then for all i we have $Y_{u,i} = s_\kappa \, E_{u,i}$ on $[0, t_0]$, where $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$. Moreover, we have $N = n$.

Remark 2.2. Kasue and Kumura [27] have obtained Lemma 2.6 in the case where N is an integer, and $\kappa \leq 0$.

Wylie and Yeroshkin [55] have proved a Laplacian comparison inequality for $N \in (-\infty, 1]$ (see Theorem 4.4 in [55]). In our case, the inequality holds as follows:

Lemma 2.7 ([55])**.** *Take $x \in \operatorname{Int} M$ and $u \in U_xM$. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa \, e^{\frac{-4f}{n-1}}$. Then for all $t \in (0, \tau_x(u))$ we have*

$$\Delta_f \, \rho_x(\gamma_u(t)) \geq H_{n,\kappa}(s_{f,u}(t)) \, e^{\frac{-2f(\gamma_u(t))}{n-1}}, \tag{2.7}$$

where $s_{f,u}$ denotes the function defined as (1.8)*.*

Wylie and Yeroshkin [55] have also shown another Laplacian comparison inequality in the case where f is bounded (see Corollary 4.11 in [55]). In our setting, by using the same method of the proof, we have the following:

Lemma 2.8 ([55])**.** *Take $x \in \operatorname{Int} M$ and $u \in U_xM$. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa \, e^{\frac{-4f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then for all $t \in (0, \tau_x(u))$ we have*

$$\Delta_f \, \rho_x(\gamma_u(t)) \geq H_{n,\kappa}(e^{-2\delta} t) \, e^{\frac{-2f(\gamma_u(t))}{n-1}}. \tag{2.8}$$

Proof. By $H'_{n,\kappa} > 0$ and $f \leq (n-1)\delta$, for every $t \in (0, \tau_x(u))$ we have $s_{f,u}(t) \geq e^{-2\delta} t$. From (2.7), it follows that

$$\Delta_f \, \rho_x(\gamma_u(t)) \geq H_{n,\kappa}(s_{f,u}(t)) \, e^{\frac{-2f(\gamma_u(t))}{n-1}} \geq H_{n,\kappa}(e^{-2\delta} t) \, e^{\frac{-2f(\gamma_u(t))}{n-1}}. \tag{2.9}$$

We obtain the desired inequality. $\square$

Wylie and Yeroshkin [55] have also proved a rigidity result in the equality case of the Laplacian comparison inequality (see Lemma 4.13 in [55]). From the argument discussed in their proof, we can conclude the following:

Lemma 2.9 ([55])**.** *Under the same setting as in Lemma 2.7, assume that for some $t_0 \in (0, \tau_x(u))$ the equality in* (2.7) *holds. Choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of T_xM with $e_{u,n} = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0_x$ and $Y'_{u,i}(0) = e_{u,i}$. Then for all i we have $Y_{u,i} = F_{\kappa,u} \, E_{u,i}$ on $[0, t_0]$, where $F_{\kappa,u}$ is the function defined as* (1.9)*, and $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_u$ is constant on $[0, t_0]$.*

Remark 2.3. Under the same setting as in Lemma 2.8, we assume that for some $t_0 \in (0, \tau_x(u))$ the equality in (2.8) holds. Then the equalities in (2.9) hold. In this case, the equality in (2.7) holds (see Lemma 2.9), and $s_{f,u}(t_0) = e^{-2\delta} t_0$; in particular, $f \circ \gamma_u = (n-1)\delta$ on $[0, t_0]$.

2.5 Warped and twisted product spaces

In order to obtain a splitting theorem of Cheeger-Gromoll type, Wylie [54] has proved that a twisted product space over $\mathbb{R}$ becomes a warped product space under a non-negativity of the 1-weighted Ricci curvature (see Proposition 2.2 in [54]). The proof is based on a pointwise calculation, and the same holds true for a twisted product space over an arbitrary interval.

By the same argument as in the proof of Proposition 2.2 in [54], we can prove the following in our setting:

Proposition 2.10 ([54]). *Let M be a Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that there exist an interval I in the form of $[0,\infty)$ or $[0,D]$ for some positive number D, and a connected component ∂M_1 of ∂M such that M is isometric to a twisted product $I \times_{F_{0,0}} \partial M_1$. If $\mathrm{Ric}^1_{f,M} \geq 0$, then there exist functions $f_0 : I \to \mathbb{R}$ and $f_1 : \partial M_1 \to \mathbb{R}$ such that for all $t \in I$ and $z \in \partial M_1$ we have $f(\gamma_z(t)) = f_0(t) + f_1(z)$; in particular, M is isometric to a warped product $\left(I \times \partial M_1, dt^2 + e^{\frac{2f_0(t)}{n-1}} h_0\right)$, where we put $h_0 := e^{2\frac{f_1 - (f|_{\partial M_1})}{n-1}} h$.*

Proposition 2.10 has been implicitly used in the proof of Theorem 5.1 in [54].

Chapter 3

Cut locus for the boundary

Throughout this chapter, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. We study basic properties of the cut locus $\mathrm{Cut}\,\partial M$ for the boundary.

3.1 Cut values

For a point $x \in M$, we recall that a point $z \in \partial M$ is said to be a foot point on ∂M of x if we have $d_M(x, z) = \rho_{\partial M}(x)$.

We first show the following (see e.g., [45]):

Lemma 3.1. *For $x \in \mathrm{Int}\, M$, let $z \in \partial M$ be a foot point on ∂M of x. Then there exists a unique minimal geodesic $\gamma : [0, l] \to M$ from z to x such that $\gamma = \gamma_z|_{[0,l]}$, where $l = \rho_{\partial M}(x)$. In particular, $\gamma'(0) = u_z$ and $\gamma|_{(0,l]}$ lies in $\mathrm{Int}\, M$.*

Proof. Since M is complete, there exists a minimal geodesic $\gamma : [0, l] \to M$ from z to x. Since z is a foot point on ∂M of x, we see that $\gamma|_{(0,l]}$ lies in $\mathrm{Int}\, M$. By the smoothness of ∂M, there exists an open neighborhood U of ∂M such that for each $y \in U$, we have a unique foot point z_y on ∂M of y; moreover, $\gamma_{z_y}|_{[0,\rho_{\partial M}(y)]}$ is a unique minimal geodesic in M from z_y to y. If $x \in U \setminus \partial M$, then z is a unique foot point on ∂M of x, and $\gamma = \gamma_z|_{[0,l]}$. Even if $x \notin U \setminus \partial M$, then for every sufficiently small $t > 0$, we see that z is a unique foot point on ∂M of $\gamma(t)$. It follows that $\gamma = \gamma_z|_{[0,l]}$. □

We have the following relationship between the inscribed radius and the function τ defined as (1.13) (see e.g., [45]):

Lemma 3.2. $D(M, \partial M) = \sup_{z \in \partial M} \tau(z)$.

Proof. Put $t_0 := \sup_{z \in \partial M} \tau(z)$. First, we show

$$D(M, \partial M) \leq t_0. \tag{3.1}$$

Take $x \in M$, and choose a foot point z on ∂M of x. Using Lemma 3.1, we see $x = \gamma_z(\rho_{\partial M}(x))$ and $\rho_{\partial M}(x) \leq \tau(z)$. We have $\rho_{\partial M}(x) \leq t_0$, and hence (3.1).

We show the opposite of (3.1). In the case of $t_0 < \infty$, for every $\epsilon > 0$ there exists $z_\epsilon \in \partial M$ such that

$$t_0 - \epsilon < \tau(z_\epsilon) = \rho_{\partial M}(\gamma_{z_\epsilon}(\tau(z_\epsilon))) \leq D(M, \partial M);$$

in particular, we have the opposite of (3.1). In the case of $t_0 = \infty$, in a similar way, we can prove $D(M, \partial M) = \infty$. This completes the proof. $\square$

Furthermore, we have the following characterization of τ (see e.g., [45]):

Lemma 3.3. *Take $z \in \partial M$ with $\tau(z) < \infty$. Let $t_0 > 0$. Then $t_0 = \tau(z)$ if and only if $t_0 = \rho_{\partial M}(\gamma_z(t_0))$, and at least one of the following holds:*

(1) *$\gamma_z(t_0)$ is the first conjugate point of ∂M along γ_z;*

(2) *there exists a foot point $\bar{z} \in \partial M \setminus \{z\}$ on ∂M of $\gamma_z(t_0)$.*

Proof. We first assume $t_0 = \rho_{\partial M}(\gamma_z(t_0))$. Then we see $t_0 \leq \tau(z)$. If (1) holds, then we have $t_0 = \tau_1(z)$; in particular, $t_0 = \tau(z)$. Suppose that (2) holds. We prove $t_0 = \tau(z)$ by contradiction. Assume $t_0 < \tau(z)$. Take $\epsilon > 0$ such that $t_0 + \epsilon < \tau(z)$. Lemma 3.1 implies $\gamma_z(t_0) = \gamma_{\bar{z}}(t_0)$. In the case of $\gamma'_z(t_0) = -\gamma'_{\bar{z}}(t_0)$, we have $\gamma_z(t_0 + \epsilon) = \gamma_{\bar{z}}(t_0 - \epsilon)$. From $t_0 \leq \tau(\bar{z})$ we deduce

$$\rho_{\partial M}(\gamma_z(t_0 + \epsilon)) = \rho_{\partial M}(\gamma_{\bar{z}}(t_0 - \epsilon)) = t_0 - \epsilon.$$

This is in contradiction with $t_0 + \epsilon < \tau(z)$. In the case of $\gamma'_z(t_0) \neq -\gamma'_{\bar{z}}(t_0)$, for all $t \in (t_0, t_0 + \epsilon]$ we have

$$\rho_{\partial M}(\gamma_z(t)) \leq d_M(\gamma_z(t), \bar{z}) < d_M(\gamma_z(t), \gamma_z(t_0)) + d_M(\gamma_z(t_0), \bar{z}) = t.$$

This contradicts $t \leq t_0 + \epsilon < \tau(z)$. Therefore, we conclude $t_0 = \tau(z)$.

We next assume $t_0 = \tau(z)$. Then we see $t_0 = \rho_{\partial M}(\gamma_z(t_0))$. It suffices to show that if z is not the first conjugate point of ∂M along γ_z, then we have (2). Assume that z is not the first conjugate point of ∂M along γ_z. Take an open neighborhood U of $(z, t_0 u_z)$ in $T^{\perp}\partial M$ such that $\exp^{\perp}|_U$ is a diffeomorphism onto its image. For every sufficiently large $i \in \mathbb{N}$, put $x_i := \gamma_z(t_0 + i^{-1})$, and take a foot point z_i on ∂M of x_i. By Lemma 3.1, there exists a unique minimal geodesic $\gamma_i : [0, l_i] \to M$ from z_i to x_i such that $\gamma_i = \gamma_{z_i}|_{[0,l_i]}$, where $l_i = \rho_{\partial M}(x_i)$. Since (M, d_M) is proper, by taking a subsequence if necessary, we may assume that for some $z_0 \in \partial M$ we have $z_i \to z_0$ in ∂M. Notice that z_0 is a foot point on ∂M of $\gamma_z(t_0)$. If we suppose $z = z_0$, then for every sufficiently large i we have $(z_i, l_i\, u_{z_i}) \in U$ and $\exp^{\perp}(z, (t_0 + i^{-1})\, u_z) = \exp^{\perp}(z_i, l_i\, u_{z_i})$. The injectivity of $\exp^{\perp}|_U$ implies $t_0 + i^{-1} = l_i$. This contradicts $t_0 + i^{-1} > l_i$. Hence we obtain $z \neq z_0$. This proves the lemma. $\square$

Using Lemma 3.3, we prove the following (see e.g., [45]):

Lemma 3.4. *The function τ is continuous on ∂M.*

Proof. Let $z_i \to z$ in ∂M. First, we prove the upper semi-continuity of τ. Put $\hat{t} := \limsup_{i\to\infty} \tau(z_i)$. Suppose $\hat{t} < \infty$. Take a subsequence $\{z_j\}$ of $\{z_i\}$ with $\tau(z_j) \to \hat{t}$ as $j \to \infty$. Putting $x_j := \gamma_{z_j}(\tau(z_j))$, we see $x_j \to \gamma_z(\hat{t})$. Furthermore, for all j we have $\rho_{\partial M}(x_j) = \tau(z_j)$. Letting $j \to \infty$, we obtain $\rho_{\partial M}(\gamma_z(\hat{t})) = \hat{t}$. It follows that $\hat{t} \leq \tau(z)$. In a similar way, we see that if $\hat{t} = \infty$, then $\tau(z) = \infty$. Therefore, we have shown the upper semi-continuity.

We next prove the lower semi-continuity. Put $\check{t} := \liminf_{i\to\infty} \tau(z_i)$. We may assume $\check{t} < \infty$. Take a subsequence $\{z_j\}$ of $\{z_i\}$ with $\tau(z_j) \to \check{t}$ as $j \to \infty$. Put $x_j := \gamma_{z_j}(\tau(z_j))$. Since $\rho_{\partial M}(\gamma_z(\check{t})) = \check{t}$, we see $\check{t} \leq \tau(z)$. Lemma 3.3 tells us that for each j, at least one of the following holds:

(1) $\tau(z_j) = \tau_1(z_j)$;

(2) there exists a foot point $\bar{z}_j \in \partial M \setminus \{z_j\}$ on ∂M of x_j.

Notice that at least one of (1), (2) holds for infinitely many j. If (1) holds for infinitely many j, then the continuity of τ_1 implies $\check{t} = \tau_1(z)$, and hence $\check{t} \geq \tau(z)$. We now suppose that (2) holds for infinitely many j. By the properness of (M, d_M), for some $\bar{z} \in \partial M$, there exists a subsequence $\{\bar{z}_k\}$ of $\{\bar{z}_j\}$ such that $\bar{z}_k \to \bar{z}$ in ∂M as $k \to \infty$. We see $\gamma_z(\check{t}) = \gamma_{\bar{z}}(\check{t})$. In the case of $z \neq \bar{z}$, the point $\gamma_z(\check{t})$ has distinct two foot points $z, \bar{z}$ on ∂M, and Lemma 3.3 implies $\check{t} = \tau(z)$. In the case of $z = \bar{z}$, we can prove that $\gamma_z(\check{t})$ is a conjugate point of ∂M along γ_z. In fact, if we suppose that $\gamma_z(\check{t})$ is not a conjugate point, then there exists an open neighborhood U of $(z, \check{t}u_z)$ in $T^{\perp}\partial M$ such that $\exp^{\perp}|_U$ is a diffeomorphism onto its image. By $z = \bar{z}$, for every sufficiently large k we have $(z_k, \tau(z_k)u_{z_k}), (\bar{z}_k, \tau(z_k)u_{z_k}) \in U$. Since $z_k \neq \bar{z}_k$, this contradicts the injectivity of $\exp^{\perp}|_U$. It follows that $\gamma_z(\check{t})$ is a conjugate point of ∂M along γ_z. Hence we have $\check{t} \geq \tau_1(z)$; in particular, $\check{t} \geq \tau(z)$. We complete the proof of the lower semi-continuity of τ. $\square$

3.2 Cut locus

We put

$$TD_{\partial M} := \bigcup_{z \in \partial M} \{\, t\, u_z \in T_z^{\perp}\partial M \mid t \in [0, \tau(z))\,\},$$
$$T\mathrm{Cut}\,\partial M := \bigcup_{z \in \partial M} \{\, \tau(z)\, u_z \in T_z^{\perp}\partial M \mid \tau(z) < \infty \,\}.$$

We define $D_{\partial M} := \exp^{\perp}(TD_{\partial M})$ and $\mathrm{Cut}\,\partial M := \exp^{\perp}(T\mathrm{Cut}\,\partial M)$. We call $\mathrm{Cut}\,\partial M$ the *cut locus for the boundary* ∂M. Note that $\mathrm{Cut}\,\partial M$ is contained in $\mathrm{Int}\, M$.

We show the following (see e.g., [45]):

Lemma 3.5. *$M = D_{\partial M} \sqcup \mathrm{Cut}\,\partial M$.*

Proof. We show
$$M = D_{\partial M} \cup \mathrm{Cut}\,\partial M. \tag{3.2}$$
Take $x \in M$, and a foot point z on ∂M of x. Put $l := \rho_{\partial M}(x)$. By Lemma 3.1, we have $x = \gamma_z(l)$. Since $l \in [0, \tau(z)]$, the point x belongs to the right hand side of (3.2). This proves (3.2).

Next, we show
$$\mathrm{Cut}\,\partial M \cap D_{\partial M} = \emptyset. \tag{3.3}$$
Suppose that there exists a point x that belongs to the left hand side of (3.3). In this case, there exist z_1, $z_2 \in \partial M$ and $t_0 \in (0, \tau(z_2))$ such that $x = \gamma_{z_1}(\tau(z_1)) = \gamma_{z_2}(t_0)$. We see $t_0 = \tau(z_1)$; in particular, $z_1 \neq z_2$. It follows that x has distinct foot points z_1, z_2 on ∂M. By Lemma 3.3, we have $t_0 = \tau(z_2)$. This is a contradiction, and hence (3.3). $\square$

Using Lemma 3.5, we see the following (see e.g., [45]):

Lemma 3.6. *Suppose that ∂M is compact. Then $D(M, \partial M) < \infty$ if and only if M is compact. In particular, if M is non-compact, then there exists $z \in \partial M$ such that $\tau(z) = \infty$.*

Proof. If M is compact, then the function $\rho_{\partial M}$ attains its maximum, and hence $D(M, \partial M) < \infty$. We suppose $D(M, \partial M) < \infty$. By the compactness of ∂M, and by Lemmas 3.2 and 3.4, the set $TD_{\partial M} \cup T\mathrm{Cut}\,\partial M$ is bounded and closed in $T^{\perp}\partial M$; in particular, it is compact. Lemma 3.5 implies that M is also compact.

Now, let M be non-compact. We prove that for some $z \in \partial M$ we have $\tau(z) = \infty$ by contradiction. Suppose that for all $z \in \partial M$ we have $\tau(z) < \infty$. In this case, the compactness of ∂M and Lemma 3.4 imply that the function τ attains its maximum. By Lemma 3.2, we see $D(M, \partial M) < \infty$. From the result in the first half, we see that M is compact. This is a contradiction. $\square$

Combining Lemmas 3.3 and 3.4, we have the following (see e.g., [45]):

Lemma 3.7. *The set* $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$ *is a maximal domain in* $T^{\perp}\partial M$ *on which* $\exp^{\perp}$ *is regular and injective. In particular,* $\exp^{\perp}|_{TD_{\partial M}\setminus 0(T^{\perp}\partial M)}$ *is a diffeomorphism onto* $D_{\partial M} \setminus \partial M$.

For the distance function $\rho_{\partial M}$ from ∂M, we prove the following (see e.g., [45]):

Proposition 3.8. *The function* $\rho_{\partial M}$ *is smooth on* $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$. *Moreover, for every* $x \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$ *we have* $\nabla \rho_{\partial M}(x) = \gamma'(l)$, *where* $\gamma : [0, l] \to M$ *is the minimal geodesic from the foot point on* ∂M *of* x *to* x.

Proof. Lemma 3.5 implies $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M = D_{\partial M} \setminus \partial M$. Using Lemma 3.7, we see that for all $x \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$, we have $\rho_{\partial M}(x) = \|(\exp^{\perp})^{-1}(x)\|$. In particular, $\rho_{\partial M}$ is smooth on $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$.

Take a vector $u \in T_x M$, and a smooth curve $c : (-\epsilon, \epsilon) \to \mathrm{Int}\, M$ tangent to u at $x = c(0)$. We may assume $c(t) \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$ when $|t|$ is sufficiently small. By Lemma 3.3, there exists a unique foot point $\bar{c}(t)$ on ∂M of $c(t)$. Lemma 3.1 enables us to obtain a smooth variation of γ by taking minimal geodesics in M from $\bar{c}(t)$ to $c(t)$. Applying the first variation formula to the variation, we obtain $(\rho_{\partial M} \circ c)'(0) = g(u, \gamma'(l))$. We derive $\nabla \rho_{\partial M}(x) = \gamma'(l)$. $\square$

For the volume of the cut locus, we prove the following (see e.g., [45]):

Proposition 3.9. $\mathrm{vol}_g \,\mathrm{Cut}\,\partial M = 0$.

Proof. By Lemma 3.4 and the Fubini theorem, the graph

$$\{\, (z, \tau(z)) \mid z \in \partial M,\ \tau(z) < \infty \,\}$$

of τ is a null set of $\partial M \times [0, \infty)$. A map $\Phi : \partial M \times [0, \infty) \to T^{\perp}\partial M$ defined by $\Phi(z, t) := (z, t u_z)$ is smooth. In particular, $T\mathrm{Cut}\,\partial M$ is also a null set of $T^{\perp}\partial M$. Since $\mathrm{Cut}\,\partial M$ is contained in $\mathrm{Int}\, M$, the map $\exp^{\perp}$ is smooth on a neighborhood of $T\mathrm{Cut}\,\partial M$ in $T^{\perp}\partial M$. Hence we have $\mathrm{vol}_g \,\mathrm{Cut}\,\partial M = 0$. $\square$

3.3 Avoiding the cut locus

For a subset Ω of M, we denote by $\bar{\Omega}$ the closure of Ω in M, and by $\partial\Omega$ the boundary of Ω in M. For a domain Ω in M such that $\partial\Omega$ is a smooth hypersurface in M, we denote by $\mathrm{vol}_{\partial\Omega}$ the canonical Riemannian volume measure on $\partial\Omega$.

We prove the following lemma to avoid the cut locus for the boundary:

Lemma 3.10 ([46]). *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Then there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of $\bar{\Omega}$ such that for every $k \in \mathbb{N}$, the set $\partial\Omega_k$ is a smooth hypersurface in M except for a null set in $(\partial\Omega, \mathrm{vol}_{\partial\Omega})$ satisfying the following properties:*

(1) *for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$;*

(2) $\bar{\Omega} \setminus \mathrm{Cut}\,\partial M = \bigcup_{k\in\mathbb{N}} \Omega_k$*;*

(3) *for every $k \in \mathbb{N}$, and for almost every point $x \in \partial\Omega_k \cap \partial\Omega$ in $(\partial\Omega, \mathrm{vol}_{\partial\Omega})$, there exists a unique unit outer normal vector for Ω_k at x that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at x;*

(4) *for every $k \in \mathbb{N}$, on $\partial\Omega_k \setminus \partial\Omega$, there exists a unique unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.*

Moreover, if $\bar{\Omega} = M$, then for every $k \in \mathbb{N}$, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfies $\partial\Omega_k \cap \partial M = \partial M$.

For the cut locus for a single point, a similar result to Lemma 3.10 has been already known (see e.g., Theorem 4.1 in [9]). We will prove Lemma 3.10 by a similar method to the case of the cut locus for a single point.

For $\Omega \subset M$, we say that $z \in \partial M$ is a *foot point on ∂M of Ω* if there exists $x \in \Omega$ such that z is a foot point on ∂M of x. We denote by $\Pi(\Omega)$ the set of all foot points on ∂M of Ω. We put

$$\Pi(\Omega)_\infty := \{\, z \in \Pi(\Omega) \mid \tau(z) = \infty \,\}, \quad \Pi(\Omega)_0 := \Pi(\Omega) \setminus \Pi(\Omega)_\infty.$$

Let $d_{\partial M}$ denote the Riemannian distance on ∂M. For $A \subset \partial M$ and $r > 0$, if $A \neq \emptyset$, then we denote by $U^{\partial M}_r(A)$ the set of all points $z \in \partial M$ such that $d_{\partial M}(z, A) < r$, and if $A = \emptyset$, then $U^{\partial M}_r(A) := \emptyset$. Put $\Pi(\Omega)_r := \Pi(\Omega) \setminus U^{\partial M}_r(\Pi(\Omega)_\infty)$. We denote by $\Omega_{\infty,r}$ the set of all points $x \in \Omega$ such that there exists a foot point on ∂M of x that belongs to $U^{\partial M}_r(\Pi(\Omega)_\infty)$. Note that if $\Pi(\Omega)_\infty = \emptyset$, then $\Pi(\Omega)_r = \Pi(\Omega)$ and $\Omega_{\infty,r} = \emptyset$; if $\Pi(\Omega)_0 = \emptyset$, then $\Omega \cap \mathrm{Cut}\,\partial M = \emptyset$.

For the proof of Lemma 3.10, we first show the following:

Lemma 3.11. *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. If $\Pi(\bar{\Omega})_0 \neq \emptyset$, then for every sufficiently small $r > 0$, we have $\Pi(\bar{\Omega})_r \neq \emptyset$ and $\bar{\Omega}_{\infty,r} \cap \mathrm{Cut}\,\partial M = \emptyset$.*

Proof. If $\Pi(\bar{\Omega})_\infty = \emptyset$, then for every r we have $\Pi(\bar{\Omega})_r = \Pi(\bar{\Omega})$ and $\bar{\Omega}_{\infty,r} = \emptyset$, and hence we complete the proof. Assume $\Pi(\bar{\Omega})_\infty \neq \emptyset$. Since $\Pi(\bar{\Omega})_0 \neq \emptyset$, Lemma 3.4 implies that for every sufficiently small r, we have $\Pi(\bar{\Omega})_r \neq \emptyset$. We prove by contradiction that for every sufficiently small r, we have $\bar{\Omega}_{\infty,r} \cap \mathrm{Cut}\,\partial M = \emptyset$. Suppose that for every $k \in \mathbb{N}$, we have $r_k \in (0, k^{-1})$ such that $\bar{\Omega}_{\infty,r_k} \cap \mathrm{Cut}\,\partial M \neq \emptyset$. For each $k \in \mathbb{N}$, take $x_k \in \bar{\Omega}_{\infty,r_k} \cap \mathrm{Cut}\,\partial M$. We may assume that for some $x \in \bar{\Omega}$, we have $x_k \to x$ in M. In this case, $x \in \mathrm{Cut}\,\partial M$. Take a foot point z_k on ∂M of x_k satisfying $z_k \in U^{\partial M}_{r_k}(\Pi(\bar{\Omega})_\infty)$. We may assume that for some $z \in \Pi(\bar{\Omega})$ we have $z_k \to z$ in ∂M. We see that z is a foot point on ∂M of x. Since $\Pi(\bar{\Omega})_\infty$ is closed in ∂M, we have $z \in \Pi(\bar{\Omega})_\infty$; in particular, $x \notin \mathrm{Cut}\,\partial M$. This is a contradiction. □

Next, we show the following:

Lemma 3.12. *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Assume that for some $r > 0$ we have $\Pi(\bar{\Omega})_r \neq \emptyset$. Then there exists an open subset U_r of ∂M containing $\Pi(\bar{\Omega})_r$ such that τ is finite on U_r, and for every $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z\in U_r} \tau(z))$ there exists a smooth function $\tau_{r,k} : U_r \to \mathbb{R}$ such that for all $z \in U_r$ we have*

$$\tau_{r,k}(z) \in \left(\tau(z) - \frac{3k+2}{3k(k+1)}, \tau(z) - \frac{3k+1}{3k(k+1)} \right).$$

Proof. By Lemma 3.4, there exists $\alpha \in (0, r)$ such that τ is finite on $U^{\partial M}_{\alpha}(\Pi(\bar{\Omega})_r)$. Put $\tilde{U}_r := U^{\partial M}_{\alpha/2}(\Pi(\bar{\Omega})_r)$. Let $\eta : [0, \infty) \to [0, \infty)$ be a smooth function such that

(1) $\eta|_{[0,1/2]}$ is constant;

(2) the support of η is contained in $[0, 1]$;

(3) for the volume ω_{n-2} of $\mathbb{S}^{n-2}$, we have

$$\omega_{n-2} \int_0^{\infty} \eta(t)\, t^{n-2}\, dt = 1.$$

For $\beta > 0$, we put $\eta_\beta(t) := \beta^{1-n}\, \eta(\beta^{-1}t)$. For $\beta \in (0, \alpha/2)$, we define a function $\tilde{\tau}_\beta : \tilde{U}_r \to \mathbb{R}$ by

$$\tilde{\tau}_\beta(z) := \int_{U^{\partial M}_{\alpha}(\Pi(\bar{\Omega})_r)} \tau(\bar{z})\, \eta_\beta(d_{\partial M}(z, \bar{z}))\, d\,\mathrm{vol}_h(\bar{z}).$$

Note that if β is smaller than the infimum of the injectivity radius of ∂M on the closure of $\tilde{U}_r$ in ∂M, then $\tilde{\tau}_\beta$ is smooth on $\tilde{U}_r$.

Fix $\beta \in (0, \alpha/2)$ such that $\tilde{\tau}_\beta$ is smooth on $\tilde{U}_r$. Take $z \in \tilde{U}_r$. By the Fubini theorem and the change of variable,

$$\int_{U^{\partial M}_{\alpha}(\Pi(\bar{\Omega})_r)} \eta_\beta(d_{\partial M}(z, \bar{z}))\, d\,\mathrm{vol}_h(\bar{z}) = \int_{U_z\partial M} \int_0^1 \eta(t)\, t^{n-2}\, \frac{\theta_{\partial M}(\beta t, u)}{(\beta t)^{n-2}}\, dt\, du,$$

where $U_z\partial M$ is the unit tangent sphere at z on ∂M, and $\theta_{\partial M}(\beta t, u)$ is the absolute value of the Jacobian of the exponential map on $T_z\partial M$ at $(\beta t)u$. The right hand side tends to 1 as $\beta \to 0$. For every sufficiently small β, we have

$$\begin{aligned} |\tilde{\tau}_\beta(z) - \tau(z)| \leq \tau(z) &\left| 1 - \int_{U^{\partial M}_{\alpha}(\Pi(\bar{\Omega})_r)} \eta_\beta(d_{\partial M}(z, \bar{z}))\, d\,\mathrm{vol}_h(\bar{z}) \right| \\ &+ \sup_{\bar{z}\in U^{\partial M}_{\beta}(z)} |\tau(z) - \tau(\bar{z})| \int_{U^{\partial M}_{\alpha}(\Pi(\bar{\Omega})_r)} \eta_\beta(d_{\partial M}(z, \bar{z}))\, d\,\mathrm{vol}_h(\bar{z}). \end{aligned}$$

The continuity of τ implies that $\tilde{\tau}_\beta(z)$ tends to $\tau(z)$ as $\beta \to 0$.

Put $U_r := U^{\partial M}_{\alpha/4}(\Pi(\bar{\Omega})_r)$. Fix $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z\in U_r} \tau(z))$. There exists $\beta_k > 0$ such that for all $\beta \in (0, \beta_k)$ and $z \in U_r$, the function $\tilde{\tau}_\beta$ is smooth on U_r, and satisfies $|\tilde{\tau}_\beta(z) - \tau(z)| < (6k(k+1))^{-1}$. Fix $\beta \in (0, \beta_k)$. Define a function $\tau_{r,k} : U_r \to \mathbb{R}$ by

$$\tau_{r,k}(z) := \tilde{\tau}_\beta(z) - \frac{2k+1}{2k(k+1)}.$$

This is a desired one. We complete the proof. $\square$

For $i = 1, 2$, let M_i be smooth manifolds (without boundary). For an open interval $I \subset \mathbb{R}$, let $\Phi : I \times M_1 \to M_2$ be a smooth map. For $t \in I$, define a map $\Phi_t : M_1 \to M_2$ by $\Phi_t(x) := \Phi(t, x)$. A transversality theorem (see e.g., [19]) implies that if Φ is transversal to a submanifold $\widetilde{M}_2$ in M_2, then for almost every $t \in I$ the map Φ_t is transversal to $\widetilde{M}_2$.

From Lemma 3.12 we deduce the following:

Lemma 3.13. *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Assume that for some $r > 0$ we have $\Pi(\bar{\Omega})_r \neq \emptyset$. Then there exists an open subset U_r of ∂M containing $\Pi(\bar{\Omega})_r$ such that τ is finite on U_r, and for every $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z \in U_r} \tau(z))$ there exists a smooth function $\tau_{r,k} : U_r \to \mathbb{R}$ such that for all $z \in U_r$ we have $\tau_{r,k}(z) \in (\tau(z) - k^{-1}, \tau(z) - (k+1)^{-1})$. Moreover, if the intersection of the set $\{\gamma_z(\tau_{r,k}(z)) \mid z \in U_r\}$ and $\partial\Omega$ is non-empty, then they intersect with each other transversally.*

Proof. By Lemma 3.12, there exists an open subset U_r of ∂M containing $\Pi(\bar{\Omega})_r$ such that τ is finite on U_r, and for every $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z \in U_r} \tau(z))$ there exists a smooth function $\tilde{\tau}_{r,k} : U_r \to \mathbb{R}$ such that for all $z \in U_r$ we have

$$\tilde{\tau}_{r,k}(z) \in \left(\tau(z) - \frac{3k+2}{3k(k+1)}, \tau(z) - \frac{3k+1}{3k(k+1)} \right).$$

For all $z \in U_r$, we see $\tilde{\tau}_{r,k}(z) \in (\tau(z) - k^{-1}, \tau(z) - (k+1)^{-1})$.

Put $\tilde{B}_{r,k} := \{\gamma_z(\tilde{\tau}_{r,k}(z)) \mid z \in U_r\}$. If $\tilde{B}_{r,k} \cap \partial\Omega = \emptyset$, then a function $\tau_{r,k}$ on U_r defined by $\tau_{r,k} := \tilde{\tau}_{r,k}$ is a desired one. We assume that $\tilde{B}_{r,k} \cap \partial\Omega \neq \emptyset$. Put

$$I_k := \left(-\frac{1}{3k(k+1)}, \frac{1}{3k(k+1)} \right), \tag{3.4}$$

and define a map $\Phi_{r,k} : I_k \times U_r \to M$ by

$$\Phi_{r,k}(t, z) := \exp^{\perp}(z, (\tilde{\tau}_{r,k}(z) + t)u_z).$$

The map $\Phi_{r,k}$ is diffeomorphic on $I_k \times U_r$; in particular, $\Phi_{r,k}(I_k \times U_r)$ is transversal to $\partial\Omega$. For $t \in I_k$, define a map $\Phi_{r,k,t} : U_r \to M$ by $\Phi_{r,k,t}(z) := \Phi_{r,k}(t, z)$. We see $\Phi_{r,k,0}(U_r) = \tilde{B}_{r,k}$. By a transversality theorem, for almost every $t \in I_k$, the set $\Phi_{r,k,t}(U_r)$ and $\partial\Omega$ intersect with each other transversally. We fix $t_0 \in I_k$ satisfying that $\Phi_{r,k,t_0}(U_r)$ and $\partial\Omega$ intersect with each other transversally, and define a function $\tau_{r,k} : U_r \to \mathbb{R}$ by $\tau_{r,k}(z) := \tilde{\tau}_{r,k}(z) + t_0$. In this case, for all $z \in U_r$ we have $\tau_{r,k}(z) \in (\tau(z) - k^{-1}, \tau(z) - (k+1)^{-1})$, and hence $\tau_{r,k}$ is a desired one. $\square$

Furthermore, we show the following:

Lemma 3.14. *Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Assume that there exists $r_0 > 0$ such that for all $r \in (0, r_0)$, we have $\Pi(\bar{\Omega})_r \neq \emptyset$ and $\bar{\Omega}_{\infty,r} \cap \operatorname{Cut} \partial M = \emptyset$. Assume further that for a fixed number $r \in (0, r_0)$, and for every $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z \in \Pi(\bar{\Omega})_r} \tau(z))$, there exists a function $\tau_{r,k} : \Pi(\bar{\Omega})_r \to \mathbb{R}$ such that for all $z \in \Pi(\bar{\Omega})_r$ we have $\tau_{r,k}(z) \in (\tau(z) - k^{-1}, \tau(z))$. Put*

$$B_{r,k} := \left\{ \gamma_x(\tau_{r,k}(z)) \mid z \in \Pi(\bar{\Omega})_r \right\}.$$

Then there exists $k_0 \in \mathbb{N}$ with $k_0^{-1} \in (0, \inf_{z \in \Pi(\bar{\Omega})_r} \tau(z))$ such that for every $k \geq k_0$, the closure of $\bar{\Omega}_{\infty,r}$ in M and $B_{r,k}$ are disjoint.

Proof. The proof is by contradiction. Suppose that there exists a sequence $\{k_i\}$ with $k_i \to \infty$ such that for each i we have $x_i \in \bar{\Omega}_{\infty,r} \cap B_{r,k_i}$. Take $z_i \in \Pi(\Omega)_r$ such that $x_i = \gamma_{z_i}(\tau_{r,k_i}(z_i))$. Since $\tau_{r,k_i}(z_i)$ is smaller than $\tau(z_i)$, we have $x_i \notin \operatorname{Cut}\partial M$; Lemma 3.3 tells us that z_i is a unique foot point on ∂M of x_i. By the definition of $\bar{\Omega}_{\infty,r}$, we have $x_i \in \partial\bar{\Omega}_{\infty,r}$. We may assume that for some $x \in \bar{\Omega}$, we have $x_i \to x$ in M. Notice that $x \in \partial\bar{\Omega}_{\infty,r}$. Since $r \in (0, r_0)$, we see $x \notin \operatorname{Cut}\partial M$. Hence for the foot point z on ∂M of x, the sequence $\{z_i\}$ converges to z in ∂M. It holds that $\rho_{\partial M}(x_i) > \tau(z_i) - k_i^{-1}$. Letting $i \to \infty$, we obtain $\rho_{\partial M}(x) = \tau(z)$. This contradicts $x \notin \operatorname{Cut}\partial M$. $\square$

Now, we prove Lemma 3.10:

Proof of Lemma 3.10. Let Ω be a domain in M such that $\partial\Omega$ is a smooth hypersurface in M. First, we assume $\Pi(\bar{\Omega})_0 \neq \emptyset$. By Lemma 3.11, for every sufficiently small r, we have $\Pi(\bar{\Omega})_r \neq \emptyset$ and $\bar{\Omega}_{\infty,r} \cap \operatorname{Cut}\partial M = \emptyset$. Fix such r. By Lemma 3.13, there exists an open subset U_r of ∂M containing $\Pi(\bar{\Omega})_r$ such that τ is finite on U_r, and for every $k \in \mathbb{N}$ with $k^{-1} \in (0, \inf_{z \in U_r} \tau(z))$ there exists a smooth function $\tau_{r,k} : U_r \to \mathbb{R}$ such that for all $z \in U_r$ we have $\tau_{r,k}(z) \in (\tau(z) - k^{-1}, \tau(z) - (k+1)^{-1})$. Moreover, if the intersection of the set $\{\gamma_z(\tau_{r,k}(z)) \mid z \in U_r\}$ and $\partial\Omega$ is non-empty, then they intersect with each other transversally. For each k, put

$$C_{r,k} := \left\{ \gamma_z(t) \mid t \in [0, \tau_{r,k}(z)), z \in \Pi(\bar{\Omega})_r \right\},$$
$$B_{r,k} := \left\{ \gamma_z(\tau_{r,k}(z)) \mid z \in \Pi(\bar{\Omega})_r \right\}.$$

By Lemma 3.14, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, the closure of $\bar{\Omega}_{\infty,r}$ in M and $B_{r,k}$ are disjoint. Define a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of compact subsets of $\bar{\Omega}$ by

$$\Omega_k := \left((C_{r,k+k_0} \cup B_{r,k+k_0}) \cap \bar{\Omega} \right) \cup \bar{\Omega}_{\infty,r}.$$

We prove that $\{\Omega_k\}_{k\in\mathbb{N}}$ is a desired sequence satisfying (1)–(4) in Lemma 3.10. To do this, we put $\tau_k := \tau_{r,k+k_0}, B_k := B_{r,k+k_0}$ and $C_k := C_{r,k+k_0}$.

For all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\tau_{k_1} < \tau_{k_2}$ on $\Pi(\bar{\Omega})_r$. This implies (1). Take $x \in \bar{\Omega} \setminus \operatorname{Cut}\partial M$ and a unique foot point z on ∂M of x. From Lemma 3.1 we derive $x = \gamma_z(\rho_{\partial M}(x))$. If $x \in U_r^{\partial M}(\Pi(\bar{\Omega})_\infty)$, then x belongs to $\bar{\Omega}_{\infty,r}$. If we have $z \in \Pi(\bar{\Omega})_r$, then for every sufficiently large k we have $\rho_{\partial M}(x) \in (0, \tau_k(z))$. Hence $x \in \Omega_k$. This implies (2).

We next prove that $\{\Omega_k\}_{k\in\mathbb{N}}$ satisfies (3). The set $\partial\Omega_k \cap \partial\Omega$ coincides with the union of $C_k \cap \partial\Omega$, $B_k \cap \partial\Omega$ and $\bar{\Omega}_{\infty,r} \cap \partial\Omega$. Note that if $\bar{\Omega} = M$, then we see $C_k \cap \partial\Omega = \partial\Omega, B_k \cap \partial\Omega = \emptyset$ and $\bar{\Omega}_{\infty,r} = \emptyset$. On the union of $C_k \cap \partial\Omega$ and $\bar{\Omega}_{\infty,r} \cap \partial\Omega$ that is a smooth hypersurface in M, there exists the unit outer normal vector field for Ω_k that coincides with the unit outer normal vector on $\partial\Omega$ for Ω. We assume $B_k \cap \partial\Omega \neq \emptyset$. Now, the set $\{\gamma_z(\tau_{r,k}(z)) \mid z \in U_r\}$ and $\partial\Omega$ intersect with each other transversally. Hence their intersection is an $(n-2)$-dimensional submanifold in $\partial\Omega$; in particular, $B_k \cap \partial\Omega$ is a null set in $(\partial\Omega, \mathrm{vol}_{\partial\Omega})$. This implies (3).

We prove that $\{\Omega_k\}_{k\in\mathbb{N}}$ satisfies (4). The set $\partial\Omega_k \setminus \partial\Omega$ coincides with the union of $B_k \cap \Omega$ and $(M \setminus C_k) \cap \partial\Omega_{\infty,r} \cap \Omega$. Since the closure of $\bar{\Omega}_{\infty,r}$ in M and B_k are disjoint, $\partial\Omega_{\infty,r}$ is contained in $C_k \cap \bar{\Omega}$. Hence $\partial\Omega_k \setminus \partial\Omega = B_k \cap \Omega$. By the smoothness of τ_k, and by $B_k \cap \operatorname{Cut}\partial M = \emptyset$, the set $\partial\Omega_k \setminus \partial\Omega$ is a smooth hypersurface in M; in particular, there exists the unit outer normal vector field $\nu_{\Omega,k}$ on $\partial\Omega_k \setminus \partial\Omega$ for

Ω_k. Let $\Pi_{\Omega,k}$ be the set of all points $z \in \Pi(\bar{\Omega})$ such that there exists $x \in B_k \cap \Omega$ of which z is a foot point on ∂M. Put

$$\tilde{C}_{\Omega,k} := \{\, (z, t\, u_z) \mid t \in [0, \tau_k(z)),\ z \in \Pi_{\Omega,k} \,\},$$
$$\tilde{B}_{\Omega,k} := \{\, (z, \tau_k(z)\, u_z) \mid z \in \Pi_{\Omega,k} \,\}.$$

Note that $B_k \cap \Omega = \exp^{\perp}(\tilde{B}_{\Omega,k})$, and $\tilde{B}_{\Omega,k}$ can be identified by the graph of τ_k in $\Pi_{\Omega,k} \times (0,\infty)$. On $\tilde{B}_{\Omega,k}$, we have the unit outer normal vector field $\tilde{\nu}_{\Omega,k}$ for $\tilde{C}_{\Omega,k}$. Let $\tilde{\nu}^{\perp}_{\Omega,k}$ denote the $T^{\perp}\partial M$-component of $\tilde{\nu}_{\Omega,k}$, and let $u_{\partial M}$ the unit inner normal vector field on ∂M. Take $x \in B_k \cap \Omega$, and a unique foot point $z \in \Pi_{\Omega,k}$ on ∂M of x. Lemma 2.3 implies that $g(\nu_{\Omega,k}, \nabla \rho_{\partial M})(x)$ is equal to $g(\tilde{\nu}^{\perp}_{\Omega,k}, u_{\partial M})(z)$ that is greater than or equal to 0. This implies (4). We complete the proof of Lemma 3.10 in the case of $\Pi(\bar{\Omega})_0 \neq \emptyset$.

Next, we assume $\Pi(\bar{\Omega})_0 = \emptyset$. Define a sequence $\{\Omega\}_{k\in\mathbb{N}}$ by $\Omega_k := \bar{\Omega}$. The definition of $\{\Omega\}_{k\in\mathbb{N}}$ implies (1). Since $\Pi(\bar{\Omega})_0 = \emptyset$, we have $\bar{\Omega} \cap \mathrm{Cut}\, \partial M = \emptyset$. It follows that $\bar{\Omega} \setminus \mathrm{Cut}\, \partial M = \bar{\Omega}$. We see $\bar{\Omega} = \bigcup_{k\in\mathbb{N}} \Omega_k$. This implies (2). For every $k \in \mathbb{N}$, we have $\partial\Omega_k = \partial\Omega$; in particular, $\partial\Omega_k \cap \partial\Omega = \partial\Omega$ and $\partial\Omega_k \setminus \partial\Omega = \emptyset$. This implies (3) and (4).

This completes the proof of Lemma 3.10. $\square$

Chapter 4

Laplacian comparisons

In what follows, let M denote an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g, and let $f : M \to \mathbb{R}$ denote a smooth function. In this chapter, we study pointwise Laplacian comparisons for the distance function $\rho_{\partial M}$ from the boundary.

4.1 Riccati inequalities

For $N \in [n, \infty)$, we have the following inequality of Riccati type:

Lemma 4.1 ([46]). *Take $z \in \partial M$. Let $N \in [n, \infty)$. Then for all $t \in (0, \tau(z))$*

$$((\Delta_f \rho_{\partial M})(\gamma_z(t)))' \geq \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{((\Delta_f \rho_{\partial M})(\gamma_z(t)))^2}{N-1}. \tag{4.1}$$

Proof. By Proposition 3.8, the function $\rho_{\partial M} \circ \gamma_z$ is smooth on $(0, \tau(z))$. We put $f_z := f \circ \gamma_z$ and $h_{f,z} := (\Delta_f \rho_{\partial M}) \circ \gamma_z$. We first assume $N \in (n, \infty)$. Applying Proposition 2.4 to the function $\rho_{\partial M}$, we have

$$\begin{aligned} 0 &= \mathrm{Ric}_f^\infty(\gamma_z'(t)) + \|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) - g(\nabla \Delta_f \rho_{\partial M}, \nabla \rho_{\partial M})(\gamma_z(t)) \\ &= \left(\mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{f_z'(t)^2}{N-n}\right) + \|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) - h_{f,z}'(t). \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) \geq \frac{(\Delta \rho_{\partial M}(\gamma_z(t)))^2}{n-1} = \frac{(h_{f,z}(t) - f_z'(t))^2}{n-1}. \tag{4.2}$$

Using (4.2) and $N \in (n, \infty)$, we see

$$\begin{aligned} 0 &\geq \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{f_z'(t)^2}{N-n} + \frac{(h_{f,z}(t) - f_z'(t))^2}{n-1} - h_{f,z}'(t) \qquad (4.3) \\ &= \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{N-1} - h_{f,z}'(t) \\ &\qquad + \frac{N-n}{(N-1)(n-1)}\left(h_{f,z}(t) - \frac{(N-1)f_z'(t)}{N-n}\right)^2 \\ &\geq \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{N-1} - h_{f,z}'(t). \end{aligned}$$

We have the desired inequality.

Next, we assume $N = n$. If f is not constant, then $\mathrm{Ric}^n_f = -\infty$. Hence it suffices to show in the case where f is constant. In this case, we have $f'_z(t) = 0$ and $\mathrm{Ric}^\infty_f(\gamma'_z(t)) = \mathrm{Ric}^n_f(\gamma'_z(t))$. By applying Proposition 2.4 to $\rho_{\partial M}$, and by (4.2),

$$\begin{aligned} 0 &= \mathrm{Ric}^n_f(\gamma'_z(t)) + \|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) - h'_{f,z}(t) \\ &\geq \mathrm{Ric}^n_f(\gamma'_z(t)) + \frac{h^2_{f,z}(t)}{n-1} - h'_{f,z}(t). \end{aligned}$$

Therefore, we obtain (4.1). $\square$

Remark 4.1. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.1) holds. Then the equality in the Cauchy-Schwarz inequality in (4.2) holds; in particular, there exists a constant c such that $\operatorname{Hess}\rho_{\partial M} = c\, g$ on the orthogonal complement of $\nabla\rho_{\partial M}$ at $\gamma_z(t_0)$. Moreover, if $N \in (n, \infty)$, then the equalities in (4.3) hold, and if $N = n$, then $f'_z(t_0) = 0$; in particular, $(N-n)h_{f,z}(t_0) = (N-1)f'_z(t_0)$.

For $N \in (-\infty, 1]$, we have the following:

Lemma 4.2 ([48])**.** *Take $z \in \partial M$. Let $N \in (-\infty, 1]$. Then for all $t \in (0, \tau(z))$*

$$\begin{aligned} &\left(\left(e^{\frac{2f}{n-1}}\,\Delta_f\rho_{\partial M}\right)(\gamma_z(t))\right)' \\ &\qquad \geq \mathrm{Ric}^N_f(\gamma'_z(t))\, e^{\frac{2f(\gamma_z(t))}{n-1}} + \frac{\left(\left(e^{\frac{2f}{n-1}}\,\Delta_f\rho_{\partial M}\right)(\gamma_z(t))\right)^2}{n-1}\, e^{\frac{-2f(\gamma_z(t))}{n-1}}. \end{aligned} \tag{4.4}$$

Proof. Proposition 3.8 tells us that $\rho_{\partial M}\circ\gamma_z$ is smooth on $(0, \tau(z))$. Put $f_z := f\circ\gamma_z$ and $h_{f,z} := (\Delta_f\rho_{\partial M})\circ\gamma_z$. We apply Proposition 2.4 to $\rho_{\partial M}$. We see

$$\begin{aligned} 0 &= \mathrm{Ric}^\infty_f(\gamma'_z(t)) + \|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) - g\left(\nabla\Delta_f\rho_{\partial M}, \nabla\rho_{\partial M}\right)(\gamma_z(t)) \\ &= \left(\mathrm{Ric}^N_f(\gamma'_z(t)) + \frac{f'_z(t)^2}{N-n}\right) + \|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) - h'_{f,z}(t). \end{aligned}$$

The Cauchy-Schwarz inequality implies

$$\|\operatorname{Hess}\rho_{\partial M}\|^2(\gamma_z(t)) \geq \frac{(\Delta\rho_{\partial M}(\gamma_z(t)))^2}{n-1} = \frac{(h_{f,z}(t) - f'_z(t))^2}{n-1}. \tag{4.5}$$

By (4.5), we have

$$\begin{aligned} 0 &\geq \mathrm{Ric}^N_f(\gamma'_z(t)) + \frac{f'_z(t)^2}{N-n} + \frac{(h_{f,z}(t) - f'_z(t))^2}{n-1} - h'_{f,z}(t) \\ &= \mathrm{Ric}^N_f(\gamma'_z(t)) + \frac{(1-N)\,f'_z(t)^2}{(n-1)\,(n-N)} + \frac{h^2_{f,z}(t)}{n-1} \\ &\qquad - \left(\frac{2h_{f,z}(t)\,f'_z(t)}{n-1} + h'_{f,z}(t)\right). \end{aligned} \tag{4.6}$$

The last term in the right hand side of (4.6) satisfies

$$\frac{2h_{f,z}(t)f'_z(t)}{n-1} + h'_{f,z}(t) = e^{\frac{-2f(\gamma_z(t))}{n-1}}\left(e^{\frac{2f(\gamma_z(t))}{n-1}}\,h_{f,z}(t)\right)'.$$

Put $F_z(t) := e^{\frac{2f(\gamma_z(t))}{n-1}} h_{f,z}(t)$. Since $N \in (-\infty, 1]$, we have

$$0 \geq \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{(1-N)\, f_z'(t)^2}{(n-1)\,(n-N)} + \frac{h_{f,z}^2(t)}{n-1} - e^{\frac{-2f(\gamma_z(t))}{n-1}} F_z'(t) \tag{4.7}$$
$$\geq \mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{n-1} - e^{\frac{-2f(\gamma_z(t))}{n-1}} F_z'(t).$$

It follows that

$$F_z'(t) \geq e^{\frac{2f(\gamma_z(t))}{n-1}} \left(\mathrm{Ric}_f^N(\gamma_z'(t)) + \frac{h_{f,z}^2(t)}{n-1} \right)$$
$$= \mathrm{Ric}_f^N(\gamma_z'(t))\, e^{\frac{2f(\gamma_z(t))}{n-1}} + \frac{F_z^2(t)}{n-1}\, e^{\frac{-2f(\gamma_z(t))}{n-1}}.$$

We arrive at the desired inequality (4.4). $\square$

Remark 4.2. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.4) holds. Then the equality in the Cauchy-Schwarz inequality in (4.5) holds; in particular, there exists a constant c such that $\mathrm{Hess}\, \rho_{\partial M} = c\, g$ on the orthogonal complement of $\nabla \rho_{\partial M}$ at $\gamma_z(t_0)$. Moreover, the equalities in (4.7) hold; in particular, $(1-N) f_z'(t_0) = 0$.

Remark 4.3. For the distance function from a single point, Wylie and Yeroshkin [55] have shown an inequality of Riccati type similar to (4.4) (see Lemma 4.1 in [55]).

4.2 Basic Laplacian comparisons

For every $z \in \partial M$, and for every $t \in (0, \tau(z))$, the value $\Delta \rho_{\partial M}(\gamma_z(t))$ is equal to the mean curvature $H_{z,t}$ of the t-level set of $\rho_{\partial M}$ at $\gamma_z(t)$ toward $\nabla \rho_{\partial M}$. In our weighted case, the definition of the weighted Laplacian implies the following:

Lemma 4.3. *Take $z \in \partial M$. Then for every $t \in (0, \tau(z))$, the value $h_{f,z}(t)$ is equal to the f-mean curvature $H_{f,z,t}$ of the t-level surface of $\rho_{\partial M}$ at $\gamma_z(t)$ toward $\nabla \rho_{\partial M}$ defined as*

$$H_{f,z,t} := H_{z,t} + g(\nabla f, \nabla \rho_{\partial M})(\gamma_z(t));$$

in particular, $h_{f,z}(t)$ converges to $H_{f,z}$ as $t \to 0$.

For $N \in (1, \infty)$, we define a function $H_{N,\kappa,\lambda} : [0, \bar{C}_{\kappa,\lambda}) \to \mathbb{R}$ by

$$H_{N,\kappa,\lambda}(t) := -(N-1) \frac{s_{\kappa,\lambda}'(t)}{s_{\kappa,\lambda}(t)}. \tag{4.8}$$

Notice that for every $t \in [0, \bar{C}_{\kappa,\lambda})$ we have

$$H_{N,\kappa,\lambda}'(t) = (N-1)\kappa + \frac{H_{N,\kappa,\lambda}^2(t)}{N-1}. \tag{4.9}$$

By Lemma 4.1, we have the following pointwise Laplacian comparison:

Lemma 4.4 ([46])**.** *Take $z \in \partial M$. For $N \in [n, \infty)$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (N-1)\kappa$, and suppose $H_{f,z} \geq (N-1)\lambda$. Then for all $t \in (0, \min\{\tau(z), \bar{C}_{\kappa,\lambda}\})$ we have*

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{N,\kappa,\lambda}(t). \tag{4.10}$$

Proof. We put $h_{f,z} := (\Delta_f \rho_{\partial M}) \circ \gamma_z$. By Lemma 4.1, and by the curvature assumption, for every $t \in (0, \tau(z))$

$$h'_{f,z}(t) \geq \mathrm{Ric}^N_f(\gamma'_z(t)) + \frac{h^2_{f,z}(t)}{N-1} \geq (N-1)\kappa + \frac{h^2_{f,z}(t)}{N-1}. \tag{4.11}$$

Combining (4.9) and (4.11), for all $t \in (0, \min\{\tau(z), \bar{C}_{\kappa,\lambda}\})$ we have

$$h'_{f,z}(t) - H'_{N,\kappa,\lambda}(t) \geq \frac{h^2_{f,z}(t) - H^2_{N,\kappa,\lambda}(t)}{N-1}. \tag{4.12}$$

We now define a function $G_{N,\kappa,\lambda,z} : (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\}) \to \mathbb{R}$ by

$$G_{N,\kappa,\lambda,z} := s^2_{\kappa,\lambda}\left(\hat{F}_z - H_{N,\kappa,\lambda}\right). \tag{4.13}$$

From (4.12) we derive

$$\begin{aligned} G'_{N,\kappa,\lambda,z} &= 2s_{\kappa,\lambda}s'_{\kappa,\lambda}\left(h_{f,z} - H_{N,\kappa,\lambda}\right) + s^2_{\kappa,\lambda}\left(h'_{f,z} - H'_{N,\kappa,\lambda}\right) \\ &\geq 2s_{\kappa,\lambda}s'_{\kappa,\lambda}\left(h_{f,z} - H_{N,\kappa,\lambda}\right) + s^2_{\kappa,\lambda}\frac{h^2_{f,z} - H^2_{N,\kappa,\lambda}}{N-1} \\ &= \frac{s^2_{\kappa,\lambda}}{N-1}\left(h_{f,z} - H_{N,\kappa,\lambda}\right)^2 \geq 0. \end{aligned} \tag{4.14}$$

By Lemma 4.3, we see

$$G_{N,\kappa,\lambda,z}(t) \to H_{f,z} - (N-1)\lambda.$$

as $t \to 0$. Since the value $H_{f,z} - (N-1)\lambda$ is non-negative, we conclude $G_{N,\kappa,\lambda,z} \geq 0$ on $(0, \min\{\tau(z), \bar{C}_{\kappa,\lambda}\})$. This proves (4.10). □

Remark 4.4. Assume that for some $t_0 \in (0, \min\{\tau(z), \bar{C}_{\kappa,\lambda}\})$ the equality in (4.10) holds. Then $G_{N,\kappa,\lambda,z}(t_0) = 0$, where $G_{N,\kappa,\lambda,z}$ is the function defined as (4.13). From $G_{N,\kappa,\lambda,z} \geq 0$ we deduce $G_{N,\kappa,\lambda,z} = 0$ on $[0, t_0]$; in particular, the equality in (4.10) holds on $[0, t_0]$. Since the equalities in (4.11), (4.12) and (4.14) hold, the equality in (4.1) also holds on $[0, t_0]$ (see Remark 4.1).

We recall that τ_f and $s_{f,z}$ are the functions defined as (1.14) and defined as (1.15), respectively. Let $t_{f,z} : [0, \tau_f(z)] \to [0, \tau(z)]$ be the inverse function of $s_{f,z}$.

By Lemma 4.2, we have the following:

Lemma 4.5 ([48]). *Take a point $z \in \partial M$. For $N \in (-\infty, 1]$, we suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and we suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then for all $s \in (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\})$ we have*

$$\Delta_f \rho_{\partial M}\left(\gamma_z\left(t_{f,z}(s)\right)\right) \geq H_{n,\kappa,\lambda}(s)\, e^{\frac{-2f\left(\gamma_z\left(t_{f,z}(s)\right)\right)}{n-1}}. \tag{4.15}$$

In particular, for all $t \in (0, \tau(z))$ with $s_{f,z}(t) \in (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\})$ we have

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(s_{f,z}(t))\, e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{4.16}$$

Proof. We define a function $F_z : (0, \tau(z)) \to \mathbb{R}$ by

$$F_z := \left(e^{\frac{2f}{n-1}} \Delta_f \rho_{\partial M} \right) \circ \gamma_z.$$

Furthermore, we define a function $\hat{F}_z : (0, \tau_f(z)) \to \mathbb{R}$ by

$$\hat{F}_z := F_z \circ t_{f,z}.$$

Using Lemma 4.2, for every $s \in (0, \tau_f(z))$ we see

$$\begin{aligned} \hat{F}_z'(s) &= F_z'(t_{f,z}(s))\, e^{\frac{2f\left(\gamma_z\left(t_{f,z}(s)\right)\right)}{n-1}} \\ &\geq \mathrm{Ric}_f^N(\gamma_z'(t_{f,z}(s)))\, e^{\frac{4f\left(\gamma_z\left(t_{f,z}(s)\right)\right)}{n-1}} + \frac{F_z^2(t_{f,z}(s))}{n-1} \\ &\geq (n-1)\kappa + \frac{\hat{F}_z^2(s)}{n-1}. \end{aligned} \tag{4.17}$$

From (4.9), for all $s \in (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\})$ we derive

$$\hat{F}_z'(s) - H_{n,\kappa,\lambda}'(s) \geq \frac{\hat{F}_z^2(s) - H_{n,\kappa,\lambda}^2(s)}{n-1}. \tag{4.18}$$

We define a function $G_{n,\kappa,\lambda,z} : (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\}) \to \mathbb{R}$ as

$$G_{n,\kappa,\lambda,z} := s_{\kappa,\lambda}^2 \left(\hat{F}_z - H_{n,\kappa,\lambda} \right). \tag{4.19}$$

By (4.18), we have

$$\begin{aligned} G_{n,\kappa,\lambda,z}' &= 2 s_{\kappa,\lambda} s_{\kappa,\lambda}' \left(\hat{F}_z - H_{n,\kappa,\lambda} \right) + s_{\kappa,\lambda}^2 \left(\hat{F}_z' - H_{n,\kappa,\lambda}' \right) \\ &\geq 2 s_{\kappa,\lambda} s_{\kappa,\lambda}' \left(\hat{F}_z - H_{n,\kappa,\lambda} \right) + s_{\kappa,\lambda}^2 \frac{\hat{F}_z^2 - H_{n,\kappa,\lambda}^2}{n-1} \\ &= \frac{s_{\kappa,\lambda}^2}{n-1} \left(\hat{F}_z - H_{n,\kappa,\lambda} \right)^2 \geq 0. \end{aligned} \tag{4.20}$$

Lemma 4.3 implies that $G_{n,\kappa,\lambda,z}(s)$ tends to a non-negative value $e^{\frac{2f(z)}{n-1}} H_{f,z} - (n-1)\lambda$ as $s \to 0$; in particular, $G_{n,\kappa,\lambda,z} \geq 0$ on $(0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\})$. We obtain (4.15). $\square$

Remark 4.5. Assume that for some $s_0 \in (0, \min\{\tau_f(z), \bar{C}_{\kappa,\lambda}\})$ the equality in (4.15) holds. Then $G_{n,\kappa,\lambda,z}(s_0) = 0$, where $G_{n,\kappa,\lambda,z}$ is the function defined as (4.19). This implies $G_{n,\kappa,\lambda,z} = 0$ on $[0, s_0]$; in particular, the equality in (4.15) holds on $[0, s_0]$. Since the equalities in (4.17), (4.18) and (4.20) hold, the equality in (4.4) holds on $[0, t_{f,z}(s_0)]$ (see Remark 4.2).

We say that κ and λ satisfy the *subharmonic-condition* if we have

$$\inf_{z \in \partial M} \inf_{t \in (0, \tau(z))} \kappa \int_0^t F_{0,0,z}^2(a)\, da \geq -\lambda,$$

where $F_{0,0,z}$ is the function defined as (1.16). Note that if $\kappa = 0$ and $\lambda = 0$, then they satisfy the subharmonic-condition.

If the curvatures are bounded from below by constants, then we have:

Lemma 4.6 ([47]). *Take a point $z \in \partial M$. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq \kappa$, and suppose $H_{f,z} \geq \lambda$. Then for all $t \in (0, \tau(z))$ we have*

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq F_{0,0,z}^{-2}(t) \left(\kappa \int_0^t F_{0,0,z}^2(a)\, da + \lambda \right). \tag{4.21}$$

In particular, if κ and λ satisfy the subharmonic-condition, then for all $t \in (0, \tau(z))$

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq 0. \tag{4.22}$$

Proof. We define a function $F_z : (0, \tau(z)) \to \mathbb{R}$ by

$$F_z := \left(e^{\frac{2f}{n-1}} \Delta_f \rho_{\partial M} \right) \circ \gamma_z, \tag{4.23}$$

and a function $\hat{G}_{\kappa,\lambda,z} : (0, \tau(z)) \to \mathbb{R}$ by

$$\begin{aligned} \hat{G}_{\kappa,\lambda,z}(t) &:= F_z(t) - \kappa \int_0^t e^{\frac{2f(\gamma_z(a))}{n-1}}\, da - \lambda\, e^{\frac{2f(z)}{n-1}} \\ &= e^{\frac{2f(z)}{n-1}} \left(F_{0,0,z}^2(t)\, \Delta_f \rho_{\partial M}(\gamma_z(t)) - \kappa \int_0^t F_{0,0,z}^2(a)\, da - \lambda \right). \end{aligned} \tag{4.24}$$

By Lemma 4.2, for all $t \in (0, \tau(z))$ we see

$$\begin{aligned} \hat{G}_{\kappa,\lambda,z}'(t) &= F_z'(t) - \kappa\, e^{\frac{2f(\gamma_z(t))}{n-1}} \\ &\geq \left(\mathrm{Ric}_f^N(\gamma_z'(t)) - \kappa \right) e^{\frac{2f(\gamma_z(t))}{n-1}} + \frac{F_z^2(t)}{n-1}\, e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq 0. \end{aligned} \tag{4.25}$$

Lemma 4.3 implies that letting $t \to 0$, we have

$$\hat{G}_{\kappa,\lambda,z}(t) \to e^{\frac{2f(z)}{n-1}} (H_{f,z} - \lambda);$$

in particular, $\hat{G}_{\kappa,\lambda,z} \geq 0$ on $(0, \tau(z))$. We arrive at (4.21). □

Remark 4.6. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.21) holds. Then $\hat{G}_{\kappa,\lambda,z}(t_0) = 0$, where $\hat{G}_{\kappa,\lambda,z}$ is the function defined as (4.24). This implies $\hat{G}_{\kappa,\lambda,z} = 0$ on $[0, t_0]$; in particular, the equalities in (4.25) hold on $[0, t_0]$. It follows that the equality in (4.4) holds on $[0, t_0]$ (see Remark 4.2). Moreover, $F_z = 0$ on $[0, t_0]$, and hence $(\Delta_f\, \rho_{\partial M}) \circ \gamma_z = 0$ on $[0, t_0]$, where F_z is the function defined as (4.23).

Remark 4.7. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.22) holds. In this case, the equality in (4.21) also holds (see Remark 4.6).

Remark 4.8. For manifolds with boundary of non-negative Ricci curvature, Perales [41] has shown a Laplacian comparison inequality for the distance function from the boundary in a barrier sense. We can prove that the Laplacian comparison inequalities (4.10), (4.16), (4.21) globally hold in a barrier sense.

4.3 Cut value comparisons

Lemma 4.4 leads us to the following estimate for τ:

Lemma 4.7 ([46])**.** *Take $z \in \partial M$. Let κ and λ satisfy the ball-condition. For $N \in [n, \infty)$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (N-1)\kappa$, and suppose $H_{f,z} \geq (N-1)\lambda$. Then we have $\tau(z) \leq C_{\kappa,\lambda}$.*

Proof. The proof is by contradiction. We suppose $\tau(z) > C_{\kappa,\lambda}$. By Lemma 4.4, for all $t \in (0, C_{\kappa,\lambda})$ we have

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{N,\kappa,\lambda}(t).$$

Letting $t \to C_{\kappa,\lambda}$, we see $\Delta_f \rho_{\partial M}(\gamma_z(t)) \to \infty$. Since the function $\rho_{\partial M} \circ \gamma_z$ is smooth on $(0, \tau(z))$, this is a contradiction. Hence we have $\tau(z) \leq C_{\kappa,\lambda}$. □

Remark 4.9. Lemma 4.7 enables us to restate the conclusion of Lemma 4.4 as follows: For all $t \in (0, \tau(z))$ we have (4.10).

For $\delta \in \mathbb{R}$, we define a function $\tau_\delta : \partial M \to (0, \infty]$ by $\tau_\delta := e^{-2\delta}\,\tau$. We see that if $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$, then we have $\tau_\delta(z) \leq \tau_f(z)$.

From Lemma 4.5, we conclude the following estimate for τ_f and τ_δ:

Lemma 4.8 ([48])**.** *Take a point $z \in \partial M$. Let us assume that κ and λ satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then we have $\tau_f(z) \leq C_{\kappa,\lambda}$. In particular, if we assume further that for $\delta \in \mathbb{R}$ we have $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$, then $\tau_\delta(z) \leq C_{\kappa,\lambda}$.*

Proof. We prove by contradiction. If we suppose $\tau_f(z) > C_{\kappa,\lambda}$, then we see that $\tau(z) > t_{f,z}(C_{\kappa,\lambda})$. From Lemma 4.5, for all $t \in (0, t_{f,z}(C_{\kappa,\lambda}))$ we deduce

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(s_{f,z}(t))\, e^{\frac{-2f(\gamma_z(t))}{n-1}};$$

in particular, $\Delta_f \rho_{\partial M}(\gamma_z(t)) \to \infty$ as $t \to t_{f,z}(C_{\kappa,\lambda})$. This contradicts the smoothness of $\rho_{\partial M} \circ \gamma_z$ on $(0, \tau(z))$. □

Remark 4.10. By using Lemma 4.8, we can restate the conclusion of Lemma 4.5 as follows: For all $s \in (0, \tau_f(z))$ we have (4.15). In particular, for all $t \in (0, \tau(z))$ we have (4.16).

4.4 Equality cases

We recall the following radial curvature equation (see e.g., Theorem 2 in [42]):

Lemma 4.9. *Let ρ be a smooth function defined on a domain in M such that $\|\nabla\rho\| = 1$. Let X be a parallel vector field along an integral curve of $\nabla\rho$ that is orthogonal to $\nabla\rho$. Then we have*

$$g(R(X, \nabla\rho)\nabla\rho, X) = g(\nabla_{\nabla\rho} A_{\nabla\rho} X, X) - g(A_{\nabla\rho} A_{\nabla\rho} X, X),$$

where R denotes the curvature tensor induced from g, and $A_{\nabla\rho}$ denotes the shape operator of the level set of ρ toward $\nabla\rho$. In particular, if there exists a function φ defined on the domain of the integral curve such that $A_{\nabla\rho} X = -\varphi X$, then

$$g(R(X, \nabla\rho)\nabla\rho, X) = -(\varphi' + \varphi^2)\|X\|^2.$$

For the equality case of (4.10), we have the following rigidity result:

Lemma 4.10 ([46]). *Under the same setting as in Lemma 4.4, assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.10) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. Then for all i we have $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0, t_0]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$; moreover, $f \circ \gamma_z = f(z) - (N-n) \log s_{\kappa,\lambda}$ on $[0, t_0]$.*

Proof. Since the equality in (4.10) holds at t_0, the equality in (4.1) also holds on $[0, t_0]$ (see Remark 4.1). In this case, there exists a function φ on the set $\gamma_z((0, t_0))$ such that at each point on $\gamma_z((0, t_0))$ we have $\operatorname{Hess} \rho_{\partial M} = \varphi\, g$ on the orthogonal complement of $\nabla \rho_{\partial M}$ (see Remark 4.4). Put $\varphi_z := \varphi \circ \gamma_z$. For each i,

$$g(A_{\nabla \rho_{\partial M}} E_{z,i}, E_{z,i}) = -\operatorname{Hess} \rho_{\partial M}(E_{z,i}, E_{z,i}) = -\varphi_z,$$

and hence $A_{\nabla \rho_{\partial M}} E_{z,i} = -\varphi_z\, E_{z,i}$. From Lemma 4.9 we deduce

$$R(E_{z,i}, \nabla \rho_{\partial M}) \nabla \rho_{\partial M} = -(\varphi_z' + \varphi_z^2) E_{z,i}. \tag{4.26}$$

We define a function $\mathcal{F}_z : [0, t_0] \to \mathbb{R}$ by

$$\mathcal{F}_z(t) := \exp\left(\int_0^t \varphi_z(a)\, da \right).$$

By (4.26), a vector field $\tilde{Y}_{z,i}$ along $\gamma_z|_{[0,t_0]}$ defined by $\tilde{Y}_{z,i} := \mathcal{F}_z\, E_{z,i}$ is a ∂M-Jacobi field along $\gamma_z|_{[0,t_0]}$ with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. It follows that $Y_{z,i}$ coincides with $\tilde{Y}_{z,i}$ on $[0, t_0]$.

We put $f_z := f \circ \gamma_z$ and $h_{f,z} := (\Delta_f\, \rho_{\partial M}) \circ \gamma_z$. We see

$$h_{f,z} = H_{N,\kappa,\lambda}, \quad (N-n) h_{f,z} = (N-1) f_z' \tag{4.27}$$

on $[0, t_0]$; in particular, $f_z = f(z) - (N-n) \log s_{\kappa,\lambda}$ (see Remarks 4.1 and 4.4). Furthermore, the equality $\operatorname{Hess} \rho_{\partial M} = \varphi\, g$ implies that for each $t \in [0, t_0]$ we have $\Delta \rho_{\partial M}(\gamma_z(t)) = -(n-1)\varphi_z(t)$. From (4.27) we derive

$$\begin{aligned}
\varphi_z(t) &= \frac{1}{n-1} \left(-h_{f,z}(t) + f_z'(t) \right) \\
&= \frac{1}{n-1} \left(-H_{N,\kappa,\lambda}(t) + \frac{(N-n) H_{N,\kappa,\lambda}(t)}{N-1} \right) = \frac{s'_{\kappa,\lambda}(t)}{s_{\kappa,\lambda}(t)}.
\end{aligned}$$

Therefore, $\mathcal{F}_z(t) = s_{\kappa,\lambda}(t)$. We conclude $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0, t_0]$. □

For the equality case of (4.15), we have:

Lemma 4.11 ([48]). *Under the same setting as in Lemma 4.5, assume that for some for some $s_0 \in (0, \tau_f(z))$ the equality in (4.15) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. Then for all i we have $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0, t_{f,z}(s_0)]$, where $F_{\kappa,\lambda,z}$ is the function defined as (1.16), and $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0, t_{f,z}(s_0)]$.*

Proof. Put $t_0 := t_{f,z}(s_0)$. By the equality assumption, there exists a function φ on $\gamma_z((0,t_0))$ such that at each point on $\gamma_z((0,t_0))$ we have $\operatorname{Hess}\rho_{\partial M} = \varphi\, g$ on the orthogonal complement of $\nabla \rho_{\partial M}$ (see Remarks 4.2 and 4.5). Put $\varphi_z := \varphi \circ \gamma_z$. By using Lemma 4.9, for each i we see

$$R(E_{z,i}, \nabla\rho_{\partial M})\nabla\rho_{\partial M} = -(\varphi_z' + \varphi_z^2)E_{z,i}.$$

Hence if we define a function $\mathcal{F}_z : [0,t_0] \to \mathbb{R}$ as

$$\mathcal{F}_z(t) := \exp\left(\int_0^t \varphi_z(a)\, da\right),$$

then we see $Y_{z,i} = \mathcal{F}_z\, E_{z,i}$ on $[0,t_0]$.

Put $f_z := f \circ \gamma_z$ and $h_{f,z} := (\Delta_f \rho_{\partial M}) \circ \gamma_z$. We have $e^{\frac{2f_z}{n-1}}\, h_{f,z} = H_{n,\kappa,\lambda} \circ s_{f,z}$ on $[0,t_0]$ (see Remarks 4.2 and 4.5). Since $\operatorname{Hess}\rho_{\partial M} = \varphi\, g$, for each $t \in [0,t_0]$ we have $\Delta \rho_{\partial M}(\gamma_z(t)) = -(n-1)\varphi_z(t)$. It follows that

$$\varphi_z(t) = \frac{1}{n-1}\left(f_z(t) - \int_0^t e^{\frac{-2f(\gamma_z(a))}{n-1}}\, (H_{n,\kappa,\lambda} \circ s_{f,z})\,(a)\, da\right)'.$$

This implies

$$\begin{aligned}\mathcal{F}_z(t) &= \exp\left(\frac{1}{n-1}\left(f_z(t) - f(z) - \int_0^t e^{\frac{-2f(\gamma_z(a))}{n-1}}\, (H_{n,\kappa,\lambda} \circ s_{f,z})(a)\, da\right)\right)\\ &= F_{\kappa,\lambda,z}(t).\end{aligned}$$

We obtain $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0,t_0]$.

Now, $(1-N)(f_z')^2 = 0$ on $[0,t_0]$ (see Remarks 4.2 and 4.5). If $N \in (-\infty, 1)$, then f_z is constant on $[0,t_0]$. This proves the lemma. $\square$

From Lemma 4.11 we deduce the following:

Lemma 4.12 ([48])**.** *Under the same setting as in Lemma 4.5, assume that for some $t_0 \in (0,\tau(z))$ the equality in (4.16) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y_{z,i}'(0) = -A_{u_z} e_{z,i}$. Then for all i we have $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0,t_0]$, where $F_{\kappa,\lambda,z}$ is the function defined as (1.16), and $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$; moreover, if $N \in (-\infty,1)$, then $f \circ \gamma_z$ is constant on $[0,t_0]$.*

For the equality case of (4.21), we have the following:

Lemma 4.13 ([47])**.** *Under the same setting as in Lemma 4.6, assume that for some $t_0 \in (0,\tau(z))$ the equality in (4.21) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y_{z,i}'(0) = -A_{u_z} e_{z,i}$. Then for all i we have $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0,t_0]$, where $F_{0,0,z}$ is the function defined as (1.16), and $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$; moreover, if $N \in (-\infty,1)$, then $f \circ \gamma_z$ is constant on $[0,t_0]$.*

Proof. By the equality assumption, there exists a function φ on $\gamma_z((0,t_0))$ such that at each point on $\gamma_z((0,t_0))$ we see $\operatorname{Hess}\rho_{\partial M} = \varphi\, g$ on the orthogonal complement of $\nabla\rho_{\partial M}$ (see Remarks 4.2 and 4.7). Put $\varphi_z := \varphi \circ \gamma_z$. By Lemma 4.9, for each i

$$R(E_{z,i}, \nabla\rho_{\partial M})\nabla\rho_{\partial M} = -(\varphi_z' + \varphi_z^2)E_{z,i}.$$

If we define a function $\mathcal{F}_z : [0,t_0] \to \mathbb{R}$ as

$$\mathcal{F}_z(t) := \exp\left(\int_0^t \varphi_z(a)\, da\right),$$

then we see $Y_{z,i} = \mathcal{F}_z\, E_{z,i}$ on $[0,t_0]$.

We have $(\Delta_f\, \rho_{\partial M}) \circ \gamma_z = 0$ on $[0,t_0]$ (see Remark 4.7). Since $\operatorname{Hess}\rho_{\partial M} = \varphi\, g$, for each $t \in [0,t_0]$ we have $\Delta\rho_{\partial M}(\gamma_z(t)) = -(n-1)\varphi_z(t)$. If we put $f_z := f \circ \gamma_z$, then $\varphi_z(t) = (n-1)^{-1} f_z'(t)$. This implies

$$\mathcal{F}_z(t) = \exp\left(\frac{f_z(t) - f(z)}{n-1}\right) = F_{0,0,z}(t).$$

We obtain $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0,t_0]$.

We have $(1-N)(f_z')^2 = 0$ on $[0,t_0]$ (see Remarks 4.2 and 4.7). If $N \in (-\infty, 1)$, then $f_z' = 0$ on $[0,t_0]$; in particular, f_z is constant. $\square$

From Lemma 4.13 we deduce the following (see Remark 4.7):

Lemma 4.14 ([47]). *Under the same setting as in Lemma 4.6, assume that for some $t_0 \in (0, \tau(z))$ the equality in (4.22) holds. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y_{z,i}'(0) = -A_{u_z} e_{z,i}$. Then for all i we have $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0,t_0]$, where $F_{0,0,z}$ is the function defined as (1.16), and $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0,t_0]$.*

Chapter 5

Global Laplacian comparisons

In this chapter, we develop the pointwise Laplacian comparisons for the distance function from the boundary studied in the previous chapter, and establish global Laplacian comparisons in a distribution sense. We first prove a global Laplacian comparison result under the curvature bound (1.3). Next, under the curvature bound (1.4), we prove global Laplacian comparison results in the case where f is bounded from above, and in the case where f is ∂M-radial. Furthermore, we show a global Laplacian comparison result under the assumption that the N-weighted Ricci curvature is bounded from below by a constant for $N \in (-\infty, 1]$.

5.1 Usual weighted cases

Lemma 4.4 tells us the following:

Lemma 5.1 ([46])**.** *Let $p \in (1,\infty)$. Take $z \in \partial M$. For $N \in [n,\infty)$, suppose that for all $t \in (0,\tau(z))$ we have $\mathrm{Ric}^N_f(\gamma_z'(t)) \geq (N-1)\kappa$, and suppose $H_{f,z} \geq (N-1)\lambda$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, and for all $t \in (0,\tau(z))$ we have*

$$\begin{aligned} &\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right)(\gamma_z(t)) \\ &\qquad \geq -\left\{ \left(\left((\varphi')^{p-1} \right)' - H_{N,\kappa,\lambda} (\varphi')^{p-1} \right) \circ \rho_{\partial M} \right\} (\gamma_z(t)), \end{aligned} \tag{5.1}$$

where $H_{N,\kappa,\lambda}$ is the function defined as (4.8).

Proof. By straightforward computations, for all $t \in (0,\tau(z))$ we see

$$\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right)(\gamma_z(t)) = -\left((\varphi')^{p-1} \right)'(t) + \Delta_{f,2}\rho_{\partial M}(\gamma_z(t))\, \left(\varphi'\right)^{p-1}(t).$$

By Lemma 4.4, we have (5.1). □

Remark 5.1. The equality case of Lemma 5.1 results in that of Lemma 4.4 (see Lemma 4.10).

Under the curvature bound (1.3), we have the following global comparison result:

Proposition 5.2 ([46]). *Let $p \in (1,\infty)$. For $N \in [n,\infty)$, if we suppose that $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$, then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$ we have*

$$\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right) \geq -\left(\left(\left(\varphi'\right)^{p-1}\right)' - H_{N,\kappa,\lambda}\left(\varphi'\right)^{p-1}\right) \circ \rho_{\partial M}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$, we have

$$\begin{aligned}\int_M \|\nabla\left(\varphi \circ \rho_{\partial M}\right)\|^{p-2} g\left(\nabla\psi, \nabla\left(\varphi \circ \rho_{\partial M}\right)\right) dm_f \\ \geq -\int_M \psi \left\{\left(\left(\left(\varphi'\right)^{p-1}\right)' - H_{N,\kappa,\lambda}\left(\varphi'\right)^{p-1}\right) \circ \rho_{\partial M}\right\} dm_f.\end{aligned} \tag{5.2}$$

Proof. By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M such that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following properties: (1) for all k_1, k_2 with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\,\partial M = \bigcup_k \Omega_k$; (3) $\partial\Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists a unique unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

Put $\Phi := \varphi \circ \rho_{\partial M}$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\mathrm{Int}\, M$. For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial\Omega_k \setminus \partial M}}\, \mathrm{vol}_k$. By the Green formula, and by $\partial\Omega_k \cap \partial M = \partial M$,

$$\begin{aligned}&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right) dm_f \\ &\quad= \int_{\Omega_k} \left(-\psi\, g\left(\nabla\left(\|\nabla\Phi\|^{p-2}\right), \nabla\Phi\right) + \|\nabla\Phi\|^{p-2}\, \psi\, \Delta_{f,2}\Phi\right) dm_f \\ &\qquad\qquad\qquad + \int_{\partial\Omega_k \setminus \partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right) dm_{f,k} \\ &\quad= \int_{\Omega_k} \psi\, \Delta_{f,p}\Phi\, dm_f + \int_{\partial\Omega_k \setminus \partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right) dm_{f,k}.\end{aligned}$$

By Lemma 5.1 and $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$, we have

$$\begin{aligned}\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right) dm_f \\ \geq -\int_{\Omega_k} \psi \left\{\left(\left(\left(\varphi'\right)^{p-1}\right)' - H_{N,\kappa,\lambda}\left(\varphi'\right)^{p-1}\right) \circ \rho_{\partial M}\right\} dm_f.\end{aligned}$$

By letting $k \to \infty$, we obtain (5.2). $\square$

Kasue [23] has proved Proposition 5.2 when $f = 0$, $N = n$ and $p = 2$.

Remark 5.2. Assume that the equality in (5.2) holds. Then for a fixed $z \in \partial M$, and for every $t \in (0, \tau(z))$ the equality in (5.1) also holds. The equality case of Proposition 5.2 results in that of Lemma 5.1 (see Remark 5.1).

5.2 Bounded cases

For $\kappa, \lambda \in \mathbb{R}$, we say that κ and λ satisfy the *monotone-condition* if $H_{n,\kappa,\lambda} \geq 0$ and $H'_{n,\kappa,\lambda} \geq 0$ on $[0, \bar{C}_{\kappa,\lambda})$. Note that κ and λ satisfy the monotone-condition if and only if either (1) κ and λ satisfy the convex-ball-condition; or (2) $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$. For κ and λ satisfying the monotone-condition, if $\kappa = 0$ and $\lambda = 0$, then $H_{n,\kappa,\lambda} = 0$ on $[0, \infty)$; otherwise, $H_{n,\kappa,\lambda} > 0$ on $(0, \bar{C}_{\kappa,\lambda})$.

We say that κ and λ satisfy the *weakly-monotone-condition* if $H'_{n,\kappa,\lambda} \geq 0$ on $[0, \bar{C}_{\kappa,\lambda})$. We see that κ and λ satisfy the weakly-monotone-condition if and only if either (1) $\kappa \geq 0$; or (2) $\kappa < 0$ and $|\lambda| \geq \sqrt{|\kappa|}$. In particular, if κ and λ satisfy the ball-condition, then they do the weakly-monotone-condition. For κ and λ satisfying the weakly-monotone-condition, if $\kappa \leq 0$ and $|\lambda| = \sqrt{|\kappa|}$, then $H_{n,\kappa,\lambda} = (n-1)\lambda$ on $[0, \infty)$; otherwise, $H'_{n,\kappa,\lambda} > 0$ on $[0, \bar{C}_{\kappa,\lambda})$.

Lemma 4.5 implies the following:

Lemma 5.3 ([48]). *Take a point $z \in \partial M$. Let us assume that κ and λ satisfy the weakly-monotone-condition. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$. Then for all $t \in (0, \tau(z))$*

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{5.3}$$

Moreover, if κ and λ satisfy the monotone-condition, then for all $t \in (0, \tau(z))$

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{-2\delta}. \tag{5.4}$$

Proof. The boundedness of f implies that for all $t \in (0, \tau(z))$

$$s_{f,z}(t) \geq e^{-2\delta} t, \quad e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq e^{-2\delta}.$$

By Lemma 4.5 and the monotonicity of $H_{n,\kappa,\lambda}$, for all $t \in (0, \tau(z))$ we have

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(s_{f,z}(t))\, e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{\frac{-2f(\gamma_z(t))}{n-1}}. \tag{5.5}$$

Hence we arrive at (5.3). Moreover, if κ and λ satisfy the monotone-condition, then (5.3) and the positivity of $H_{\kappa,\lambda}$ imply

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{\frac{-2f(\gamma_z(t))}{n-1}} \geq H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{-2\delta}. \tag{5.6}$$

This proves the lemma. □

Remark 5.3. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (5.3) holds. Then the equalities in (5.5) hold; in particular, the equality in (4.16) holds (see Lemma 4.12). Moreover, if either (1) $\kappa > 0$; or (2) $\kappa \leq 0$ and $|\lambda| > \sqrt{|\kappa|}$, then $H'_{n,\kappa,\lambda} > 0$ on $[0, \bar{C}_{\kappa,\lambda})$, and hence $s_{f,z}(t_0) = e^{-2\delta} t_0$; in particular, $f \circ \gamma_z = (n-1)\delta$ on $[0, t_0]$.

Remark 5.4. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (5.4) holds. Then the equalities in (5.6) hold; in particular, the equality in (5.3) holds (see Remark 5.3). Moreover, if either (1) κ and λ satisfy the convex-ball-condition; or (2) $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then we have $H_{\kappa,\lambda} > 0$ on $(0, \bar{C}_{\kappa,\lambda})$, and hence $e^{-2}_{f,z}(t_0) = e^{-2\delta}$; in particular, $(f \circ \gamma_z)(t_0) = (n-1)\delta$.

From Lemma 5.3 we deduce the following:

Lemma 5.4 ([48])**.** *Let $p \in (1,\infty)$. Take $z \in \partial M$. Let κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, and for all $t \in (0, \tau(z))$ we have*

$$\begin{aligned} &\Delta_{f,p} \left(\varphi \circ \rho_{\partial M, \delta}\right)(\gamma_z(t)) \\ &\quad \geq -e^{-2p\delta} \left\{ \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} (\varphi')^{p-1} \right) \circ \rho_{\partial M,\delta} \right\} (\gamma_z(t)), \end{aligned} \tag{5.7}$$

where the function $\rho_{\partial M,\delta} : M \to \mathbb{R}$ is defined as $\rho_{\partial M,\delta} := e^{-2\delta} \rho_{\partial M}$.

Proof. Put $\Phi := \varphi \circ \rho_{\partial M,\delta}$. Define a function $\varphi_\delta : [0,\infty) \to \mathbb{R}$ by $\varphi_\delta(t) := \varphi(e^{-2\delta} t)$. We see $\Phi = \varphi_\delta \circ \rho_{\partial M}$. For each $t \in (0, \tau(z))$,

$$\Delta_{f,p} \Phi(\gamma_z(t)) = -\left((\varphi_\delta')^{p-1} \right)'(t) + \Delta_{f,2} \rho_{\partial M}(\gamma_z(t)) \, \left(\varphi_\delta'\right)^{p-1}(t).$$

By using Lemma 5.3, we have

$$\Delta_{f,p} \Phi(\gamma_z(t)) \geq -\left((\varphi_\delta')^{p-1} \right)'(t) + H_{n,\kappa,\lambda}(e^{-2\delta} t)\, e^{-2\delta} \, \left(\varphi_\delta'\right)^{p-1}(t). \tag{5.8}$$

Since

$$\begin{aligned} \left(\varphi_\delta'\right)^{p-1}(t) &= e^{-2(p-1)\delta} \, \left(\varphi'\right)^{p-1}(e^{-2\delta} t), \\ \left((\varphi_\delta')^{p-1} \right)'(t) &= -e^{-2p\delta} \left((\varphi')^{p-1} \right)'(e^{-2\delta} t), \end{aligned}$$

the right hand side of (5.8) is equal to that of (5.7). $\square$

Remark 5.5. The equality case of Lemma 5.4 results in that of Lemma 5.3 (see Remark 5.4).

Under the curvature bound (1.4), if f is bounded from above, then we have the following global Laplacian comparison result:

Proposition 5.5 ([48])**.** *Let $p \in (1,\infty)$. Let us assume that κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, we have*

$$\Delta_{f,p} \left(\varphi \circ \rho_{\partial M,\delta}\right) \geq -e^{-2p\delta} \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, \left(\varphi'\right)^{p-1} \right) \circ \rho_{\partial M,\delta}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$*, we have*

$$\begin{aligned} &\int_M \|\nabla \left(\varphi \circ \rho_{\partial M,\delta}\right)\|^{p-2} g\left(\nabla \psi, \nabla \left(\varphi \circ \rho_{\partial M,\delta}\right)\right) \, d\, m_f \\ &\quad \geq -e^{-2p\delta} \int_M \psi \, \left\{ \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, \left(\varphi'\right)^{p-1} \right) \circ \rho_{\partial M,\delta} \right\} d\, m_f. \end{aligned} \tag{5.9}$$

Proof. By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M such that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following properties: (1) for all k_1, k_2 with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\,\partial M = \bigcup_k \Omega_k$; (3) $\partial\Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists a unique unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

Put $\Phi := \varphi \circ \rho_{\partial M,\delta}$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\mathrm{Int}\, M$. For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial\Omega_k\setminus\partial M}}\, \mathrm{vol}_k$. From the Green formula we derive

$$
\begin{aligned}
&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \\
&\quad = \int_{\Omega_k} \left(-\psi\, g\left(\nabla\left(\|\nabla\Phi\|^{p-2}\right), \nabla\Phi\right) + \|\nabla\Phi\|^{p-2}\, \psi\, \Delta_{f,2}\Phi\right)\, d\,m_f \\
&\qquad\qquad\qquad + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right)\, d\,m_{f,k}.
\end{aligned}
$$

This is equal to

$$
\int_{\Omega_k} \psi\, \Delta_{f,p}\Phi\, d\,m_f + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right)\, d\,m_{f,k}.
$$

From Lemma 5.4 and $g(\nu_k, \nabla\rho_{\partial M,\delta}) \geq 0$ we deduce

$$
\begin{aligned}
&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, d\,m_f \\
&\qquad \geq -e^{-2p\delta} \int_{\Omega_k} \psi\, \left\{\left(\left(\left(\varphi'\right)^{p-1}\right)' - H_{n,\kappa,\lambda}\, \left(\varphi'\right)^{p-1}\right) \circ \rho_{\partial M,\delta}\right\}\, d\,m_f.
\end{aligned}
$$

Letting $k \to \infty$, we have the desired inequality. $\square$

Remark 5.6. Assume that the equality in (5.9) holds. In this case, for a fixed $z \in \partial M$ we see that for every $t \in (0, \tau(z))$ the equality in (5.7) also holds. The equality case of Proposition 5.5 results in that of Lemma 5.4 (see Remark 5.5).

5.3 Radial cases

We suppose that f is ∂M-radial. In this case, there exists a smooth function $\phi_f : [0,\infty) \to \mathbb{R}$ such that we have $f = \phi_f \circ \rho_{\partial M}$ on M. We define a function $s_f : [0,\infty] \to [0,\infty]$ by

$$
s_f(t) := \int_0^t e^{\frac{-2\phi_f(a)}{n-1}}\, da. \tag{5.10}
$$

Let I_f denote the image of s_f, and let $t_f : I_f \to [0,\infty]$ be the inverse function of s_f. Notice that the function $\rho_{\partial M,f}$ defined as (1.20) satisfies $\rho_{\partial M,f} = s_f \circ \rho_{\partial M}$ on M. Furthermore, for every $z \in \partial M$, we see $s_{f,z} = s_f$ and $t_{f,z} = t_f$ on $[0, \tau(z)]$ and on $[0, \tau_f(z)]$, respectively.

If f is ∂M-radial, then we see the following:

Lemma 5.6 ([48])**.** *Let us suppose that f is ∂M-radial. Let $p \in (1,\infty)$. Take a point $z \in \partial M$. For $N \in (-\infty,1]$, suppose that for all $t \in (0,\tau(z))$ we have $\mathrm{Ric}^N_f(\gamma_z'(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, and for all $t \in (0,\tau(z))$*

$$\begin{aligned}&\Delta_{\frac{n+1-2p}{n-1}f,p}\,(\varphi\circ\rho_{\partial M,f})\,(\gamma_z(t)) \qquad (5.11)\\ &\qquad\geq -e^{\frac{-2pf}{n-1}}\left\{\left(\left((\varphi')^{p-1}\right)' - H_{n,\kappa,\lambda}\,(\varphi')^{p-1}\right)\circ\rho_{\partial M,f}\right\}(\gamma_z(t)).\end{aligned}$$

Proof. Put $\Phi := \varphi\circ\rho_{\partial M,f}$ and $\varphi_f := \varphi\circ s_f$. In this case, we see $\Phi = \varphi_f\circ\rho_{\partial M}$. For each $t\in(0,\tau(z))$, the left hand side of (5.11) is equal to

$$-\left(\left(\varphi_f'\right)^{p-1}\right)'(t) + \Delta_{\frac{n+1-2p}{n-1}f,2}\,\rho_{\partial M}(\gamma_z(t))\,\left(\varphi_f'\right)^{p-1}(t).$$

We have

$$\begin{aligned}\Delta_{\frac{n+1-2p}{n-1}f,2}\,\rho_{\partial M}(\gamma_z(t)) &= \left(\Delta_f\,\rho_{\partial M} - \frac{2(p-1)}{n-1}\,g\left(\nabla f,\nabla\rho_{\partial M}\right)\right)(\gamma_z(t))\\ &= \Delta_f\,\rho_{\partial M}(\gamma_z(t)) - \frac{2(p-1)}{n-1}\,\phi_f'(t).\end{aligned}$$

By Lemma 4.5, and by $s_{f,z}(t) = s_f(t)$ and $e^{\frac{-2f(\gamma_z(t))}{n-1}} = s_f'(t)$,

$$\begin{aligned}\Delta_{\frac{n+1-2p}{n-1}f,p}\,\Phi(\gamma_z(t)) \geq &-\left(\left(\varphi_f'\right)^{p-1}\right)'(t) + H_{n,\kappa,\lambda}(s_f(t))\,s_f'(t)\,\left(\varphi_f'\right)^{p-1}(t)\\ &- \frac{2(p-1)}{n-1}\,\phi_f'(t)\,\left(\varphi_f'\right)^{p-1}(t),\end{aligned}$$

where the function s_f is defined as (5.10). Note that

$$\begin{aligned}\left(\varphi_f'\right)^{p-1}(t) &= (\varphi')^{p-1}(s_f(t))\,(s_f')^{p-1}(t),\\ \left(\left(\varphi_f'\right)^{p-1}\right)'(t) &= \left((\varphi')^{p-1}\right)'(s_f(t))\,(s_f')^p(t) + (\varphi')^{p-1}(s_f(t))\,((s_f')^{p-1})'(t)\\ &= \left((\varphi')^{p-1}\right)'(s_f(t))\,(s_f')^p(t) - \frac{2(p-1)}{n-1}\phi_f'(t)\,\left(\varphi_f'\right)^{p-1}(t).\end{aligned}$$

We also notice that $\rho_{\partial M,f} = s_f\circ\rho_{\partial M}$ on M. Therefore, we obtain

$$\begin{aligned}&\Delta_{\frac{n+1-2p}{n-1}f,p}\,\Phi(\gamma_z(t))\\ &\geq -(s_f')^p(t)\left(\left((\varphi')^{p-1}\right)'(s_f(t)) - H_{n,\kappa,\lambda}(s_f(t))\,\left(\varphi'\right)^{p-1}(s_f(t))\right)\\ &= -e^{\frac{-2pf}{n-1}}\left\{\left(\left((\varphi')^{p-1}\right)' - H_{n,\kappa,\lambda}\,(\varphi')^{p-1}\right)\circ\rho_{\partial M,f}\right\}(\gamma_z(t)).\end{aligned}$$

We conclude the lemma. $\square$

From Lemma 5.6 we derive the following:

Proposition 5.7 ([48]). *Let us suppose that f is ∂M-radial. Let $p \in (1,\infty)$. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, we have*

$$\Delta_{\frac{n+1-2p}{n-1} f,p} \left(\varphi \circ \rho_{\partial M,f}\right) \geq -e^{\frac{-2pf}{n-1}} \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, (\varphi')^{p-1} \right) \circ \rho_{\partial M,f}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$, we have

$$\int_M \|\nabla \left(\varphi \circ \rho_{\partial M,f}\right)\|^{p-2} g\left(\nabla \psi, \nabla \left(\varphi \circ \rho_{\partial M,f}\right)\right) \, d\, m_{\frac{n+1-2p}{n-1} f}$$
$$\geq -\int_M \psi \left\{ \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, (\varphi')^{p-1} \right) \circ \rho_{\partial M,f} \right\} d\, m_{\frac{n+1}{n-1} f}.$$

Proof. By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M such that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following properties: (1) for all k_1, k_2 with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\, \partial M = \bigcup_k \Omega_k$; (3) $\partial \Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists a unique unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla \rho_{\partial M}) \geq 0$.

Put $\Phi := \varphi \circ \rho_{\partial M,f}$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\mathrm{Int}\, M$. Put

$$\hat{f} := \frac{n+1-2p}{n-1} f.$$

We also put $m_{\hat{f},k} := e^{-\hat{f}|_{\partial\Omega_k \setminus \partial M}} \, \mathrm{vol}_k$, where vol_k is the canonical Riemannian volume measure on $\partial\Omega_k \setminus \partial M$. By the Green formula, and by $\partial\Omega_k \cap \partial M = \partial M$,

$$\int_{\Omega_k} \|\nabla \Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right) \, d\, m_{\hat{f}}$$
$$= \int_{\Omega_k} \left(-\psi\, g\left(\nabla \left(\|\nabla\Phi\|^{p-2}\right), \nabla\Phi\right) + \|\nabla\Phi\|^{p-2}\, \psi\, \Delta_{\hat{f},2} \Phi \right) d\, m_{\hat{f}}$$
$$+ \int_{\partial\Omega_k \setminus \partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right) \, d\, m_{\hat{f},k},$$

which is equal to

$$\int_{\Omega_k} \psi\, \Delta_{\hat{f},p} \Phi \, d\, m_{\hat{f}} + \int_{\partial\Omega_k \setminus \partial M} \|\nabla\Phi\|^{p-2}\, \psi\, g\left(\nu_k, \nabla\Phi\right) \, d\, m_{\hat{f},k}.$$

Lemma 5.6 and $g(\nu_k, \nabla \rho_{\partial M,f}) \geq 0$ imply

$$\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right) \, d\, m_{\hat{f}}$$
$$\geq -\int_{\Omega_k} \psi\, e^{\frac{-2pf}{n-1}} \left\{ \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, (\varphi')^{p-1} \right) \circ \rho_{\partial M,f} \right\} d\, m_{\hat{f}}$$
$$= -\int_{\Omega_k} \psi \left\{ \left(\left((\varphi')^{p-1} \right)' - H_{n,\kappa,\lambda} \, (\varphi')^{p-1} \right) \circ \rho_{\partial M,f} \right\} d\, m_{\frac{n+1}{n-1} f}.$$

By letting $k \to \infty$, we arrive at the desired inequality. $\square$

5.4 Subharmonic cases

If the curvatures are bounded by constants, then we see the following:

Lemma 5.8 ([47]). *Let $p \in (1,\infty)$. Take $z \in \partial M$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty,1]$, suppose that for all $t \in (0,\tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq \kappa$, and suppose $H_{f,z} \geq \lambda$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, and for all $t \in (0,\tau(z))$ we have*

$$\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right)(\gamma_z(t)) \geq -\left(\left((\varphi')^{p-1}\right)' \circ \rho_{\partial M}\right)(\gamma_z(t)). \tag{5.12}$$

Proof. By straightforward computations, for all $t \in (0,\tau(z))$ we see

$$\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right)(\gamma_z(t)) = -\left((\varphi')^{p-1}\right)'(t) + \Delta_{f,2}\rho_{\partial M}(\gamma_z(t))\ (\varphi')^{p-1}(t).$$

This together with Lemma 4.6 implies (5.12). $\square$

Remark 5.7. The equality case of Lemma 5.8 results in that of Lemma 4.6 (see Lemma 4.14).

From Lemma 5.8 we derive the following:

Proposition 5.9 ([47]). *Let $p \in (1,\infty)$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty,1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. Then for every monotone increasing smooth function $\varphi : [0,\infty) \to \mathbb{R}$, we have*

$$\Delta_{f,p}\left(\varphi \circ \rho_{\partial M}\right) \geq -\left((\varphi')^{p-1}\right)' \circ \rho_{\partial M}$$

in a distribution sense on M. More precisely, for every non-negative smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$, *we have*

$$\begin{aligned}\int_M \|\nabla\left(\varphi \circ \rho_{\partial M}\right)\|^{p-2} g\left(\nabla\psi, \nabla\left(\varphi \circ \rho_{\partial M}\right)\right) d\,m_f \qquad\qquad (5.13)\\ \geq -\int_M \psi \left(\left((\varphi')^{p-1}\right)' \circ \rho_{\partial M}\right) d\,m_f.\end{aligned}$$

Proof. By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of closed subsets of M such that for every k, the set $\partial\Omega_k$ is a smooth hypersurface in M, and satisfying the following properties: (1) for all k_1, k_2 with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $M \setminus \mathrm{Cut}\,\partial M = \bigcup_k \Omega_k$; (3) $\partial\Omega_k \cap \partial M = \partial M$ for all k; (4) for each k, on $\partial\Omega_k \setminus \partial M$, there exists a unique unit outer normal vector field ν_k for Ω_k with $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

Put $\Phi := \varphi \circ \rho_{\partial M}$. Let $\psi : M \to \mathbb{R}$ be a non-negative smooth function whose support is compact and contained in $\mathrm{Int}\, M$. For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial M$, put $m_{f,k} := e^{-f|_{\partial\Omega_k\setminus\partial M}}\, \mathrm{vol}_k$. From the Green formula we deduce

$$\begin{aligned}&\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right) d\,m_f\\ &\quad= \int_{\Omega_k} \left(-\psi\, g\left(\nabla\left(\|\nabla\Phi\|^{p-2}\right), \nabla\Phi\right) + \|\nabla\Phi\|^{p-2}\,\psi\,\Delta_{f,2}\Phi\right) d\,m_f\\ &\qquad\qquad + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\,\psi\, g\left(\nu_k, \nabla\Phi\right) d\,m_{f,k}\\ &\quad= \int_{\Omega_k} \psi\,\Delta_{f,p}\Phi\, d\,m_f + \int_{\partial\Omega_k\setminus\partial M} \|\nabla\Phi\|^{p-2}\,\psi\, g\left(\nu_k, \nabla\Phi\right) d\,m_{f,k}.\end{aligned}$$

Using Lemma 5.8 and $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$, we have

$$\int_{\Omega_k} \|\nabla\Phi\|^{p-2} g\left(\nabla\psi, \nabla\Phi\right)\, dm_f \geq -\int_{\Omega_k} \psi\, \left(\left(\left(\varphi'\right)^{p-1}\right)' \circ \rho_{\partial M}\right)\, dm_f.$$

By letting $k \to \infty$, we complete the proof. □

Remark 5.8. Assume that the equality in (5.13) holds. Then for a fixed $z \in \partial M$, and for every $t \in (0, \tau(z))$ the equality in (5.12) also holds. The equality case of Proposition 5.9 results in that of Lemma 5.8 (see Remark 5.7).

Chapter 6

Inscribed radius rigidity

In the present chapter, we prove comparison results and rigidity theorems for the inscribed radii.

6.1 Inscribed radius comparisons

We prove the following inscribed radius comparison result:

Proposition 6.1 ([46])**.** *Let κ and λ satisfy the ball-condition. For $N \in [n, \infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then we have*

$$D(M, \partial M) \leq C_{\kappa,\lambda}.$$

Proof. Lemma 3.2 implies $D(M, \partial M) = \sup_{z \in \partial M} \tau(z)$. By Lemma 4.8, we obtain the desired inequality. □

Under the curvature bound (1.4), we obtain the following:

Proposition 6.2 ([48])**.** *Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Then we have*

$$D^{g_f}(M, \partial M) \leq C_{\kappa,\lambda},$$

where $D^{g_f}(M, \partial M)$ is defined as (1.6).

Proof. Take $x \in M$, and a foot point z_x on ∂M of x. Put $l := \rho_{\partial M}(x)$. We see

$$\rho^{g_f}_{\partial M}(x) \leq L_{g_f}(\gamma_{z_x}|_{[0,l]}) = \int_0^l e^{\frac{-2f(\gamma_{z_x}(a))}{n-1}}\, da \leq \tau_f(z_x) \leq \sup_{z \in \partial M} \tau_f(z),$$

where $L_{g_f}(\gamma_{z_x}|_{[0,l]})$ denotes the length of $\gamma_{z_x}|_{[0,l]}$ determined by the Riemannian metric g_f; in particular, $D^{g_f}(M, \partial M) \leq \sup_{z \in \partial M} \tau_f(z)$. By using Lemma 4.8, we have $D^{g_f}(M, \partial M) \leq C_{\kappa,\lambda}$. This proves the comparison result. □

If f is bounded from above, then we have the following:

Proposition 6.3 ([48])**.** *Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then we have*

$$D_\delta(M, \partial M) \leq C_{\kappa,\lambda},$$

where $D_\delta(M, \partial M)$ is defined as (1.11).

Proof. We see $D_\delta(M,\partial M) = e^{-2\delta} D(M,\partial M)$. For the function $\tau_\delta : \partial M \to (0,\infty]$ defined as $\tau_\delta := e^{-2\delta}\tau$, by using Lemma 3.2, we have $D_\delta(M,\partial M) = \sup_{z\in\partial M} \tau_\delta(z)$. Hence Lemma 4.8 implies $D_\delta(M,\partial M) \le C_{\kappa,\lambda}$. This completes the proof. □

6.2 Inscribed radius rigidity

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let κ and λ satisfy the ball-condition. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \ge (N-1)\kappa$ and $H_{f,\partial M} \ge (N-1)\lambda$. By Proposition 6.1, we have the comparison inequality (1.5).

Let $x_0 \in M$ satisfy $\rho_{\partial M}(x_0) = C_{\kappa,\lambda}$. Put

$$\Omega := \{x \in \operatorname{Int} M \setminus \{x_0\} \mid \rho_{\partial M}(x) + \rho_{x_0}(x) = C_{\kappa,\lambda}\}.$$

Take a foot point z_{x_0} on ∂M of x_0, and a minimal geodesic $\gamma_0 : [0,l] \to M$ from z_{x_0} to x_0. For all $t \in (0,l)$, we have $\gamma_0(t) \in \Omega$. Hence Ω is a non-empty closed subset of $\operatorname{Int} M \setminus \{x_0\}$.

We show that Ω is open in $\operatorname{Int} M \setminus \{x_0\}$. Fix $x \in \Omega$, and take a foot point z_x on ∂M of x. Note that z_x is also a foot point on ∂M of x_0. Let $\gamma : [0,l] \to M$ be the minimal geodesic from z_x to x_0. Then $\gamma|_{(0,l)}$ passes through x. There exists an open neighborhood U of x such that ρ_{x_0} and $\rho_{\partial M}$ are smooth on U, and for every $y \in U$ there exists a unique minimal geodesic in M from x_0 to y that lies in $\operatorname{Int} M$. By Lemmas 2.6 and 4.4, for each $y \in U$

$$-\frac{\Delta_f(\rho_{\partial M} + \rho_{x_0})(y)}{N-1} \le \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M}(y)) + \frac{s'_\kappa}{s_\kappa}(\rho_{x_0}(y)) = \frac{s_{\kappa,\lambda}(\rho_{\partial M}(y) + \rho_{x_0}(y))}{s_{\kappa,\lambda}(\rho_{\partial M}(y))s_\kappa(\rho_{x_0}(y))} \le 0.$$

From Lemma 2.5 we deduce $U \subset \Omega$. We conclude that Ω is open.

Since $\operatorname{Int} M \setminus \{x_0\}$ is connected, it holds that $\Omega = \operatorname{Int} M \setminus \{x_0\}$. We have $\rho_{\partial M} + \rho_{x_0} = C_{\kappa,\lambda}$ on M. This implies $M = B_{C_{\kappa,\lambda}}(x_0)$. Furthermore, for each $u \in U_{x_0}M$, we see that $\tau_{x_0}(u) = C_{\kappa,\lambda}$, and that γ_u is orthogonal to ∂M at $C_{\kappa,\lambda}$. The equality in (2.6) holds on $\operatorname{Int} M \setminus \{x_0\}$. Choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of $T_{x_0}M$ with $e_{u,n} = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0_{x_0}$ and $Y'_{u,i}(0) = e_{u,i}$. For all i we see $Y_{u,i} = s_\kappa\, E_{u,i}$ on $[0, C_{\kappa,\lambda}]$, where $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$; moreover, $N = n$ (see Remark 2.6). We define a diffeomorphism $\Phi : [0, C_{\kappa,\lambda}] \times U_{x_0}M \to M$ by $\Phi(t,u) := \gamma_u(t)$. By the rigidity of Jacobi fields, we see that Φ is a Riemannian isometry with boundary from $B^n_{\kappa,\lambda}$ to M. Therefore, M is isometric to $B^n_{\kappa,\lambda}$. We complete the proof of Theorem 1.1. □

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \ge (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \ge (n-1)\lambda e^{\frac{-2f}{n-1}}$. Proposition 6.2 implies the comparison inequality (1.10).

Suppose that $x_0 \in M$ satisfies $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$. Put $l := \rho_{\partial M}(x_0)$ and

$$\Omega := \{x \in \operatorname{Int} M \setminus \{x_0\} \mid \rho_{\partial M}(x) + \rho_{x_0}(x) = l\}.$$

We see that Ω is a non-empty closed subset of $\mathrm{Int}\, M \setminus \{x_0\}$.

We show that Ω is open in $\mathrm{Int}\, M \setminus \{x_0\}$. Fix $x \in \Omega$, and take a foot point z_x on ∂M of x. There exists an open neighborhood U of x such that ρ_{x_0} and $\rho_{\partial M}$ are smooth on U, and for every $y \in U$ there exists a unique minimal geodesic in M from x_0 to y that lies in $\mathrm{Int}\, M$. By Lemma 2.7 and (4.16), for each $y \in U$

$$-\frac{\Delta_f(\rho_{\partial M}+\rho_{x_0})(y)}{(n-1)\, e^{\frac{-2f(y)}{n-1}}} \leq \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(s_{f,z_y}(\rho_{\partial M}(y))) + \frac{s'_{\kappa}}{s_{\kappa}}(s_{f,u_y}(\rho_{x_0}(y))) \tag{6.1}$$
$$= \frac{s_{\kappa,\lambda}(s_{f,z_y}(\rho_{\partial M}(y)) + s_{f,u_y}(\rho_{x_0}(y)))}{s_{\kappa,\lambda}(s_{f,z_u}(\rho_{\partial M}(y)))\, s_{\kappa}(s_{f,u_y}(\rho_{x_0}(y)))},$$

where z_y denotes a unique foot point on ∂M of y, and u_y denotes the initial velocity vector at x_0 of the minimal geodesic from x_0 to y. Let $\rho^{g_f}_{x_0} : M \to \mathbb{R}$ be the distance function from x_0 induced from g_f defined as $\rho^{g_f}_{x_0}(\hat{y}) := d^{g_f}_M(\hat{y}, x_0)$. From the triangle inequality for the distance function $d^{g_f}_M$, we see

$$s_{f,z_y}(\rho_{\partial M}(y)) + s_{f,u_y}(\rho_{x_0}(y)) = L_{g_f}(\gamma_{z_y}|_{[0,\rho_{\partial M}(y)]}) + L_{g_f}(\gamma_{u_y}|_{[0,\rho_{x_0}(y)]}) \tag{6.2}$$
$$\geq \rho^{g_f}_{\partial M}(y) + \rho^{g_f}_{x_0}(y) \geq \rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda},$$

where $L_{g_f}(\gamma_{z_y}|_{[0,\rho_{\partial M}(y)]})$ and $L_{g_f}(\gamma_{u_y}|_{[0,\rho_{x_0}(y)]})$ are the length of the curve $\gamma_{z_y}|_{[0,\rho_{\partial M}(y)]}$ and of the curve $\gamma_{u_y}|_{[0,\rho_{x_0}(y)]}$ determined by the Riemannian metric g_f, respectively. By combining (6.1) and (6.2), we obtain $\Delta_f(\rho_{\partial M}+\rho_{x_0})(y) \geq 0$. By Lemma 2.5, we have $U \subset \Omega$. Hence Ω is open.

Since $\mathrm{Int}\, M \setminus \{x_0\}$ is connected, we have $\Omega = \mathrm{Int}\, M \setminus \{x_0\}$. Hence $\rho_{\partial M} + \rho_{x_0} = l$ on M. This implies $M = B_l(x_0)$. For each $u \in U_{x_0}M$, we have $\tau_{x_0}(u) = l$, and γ_u is orthogonal to ∂M at l. The equality in (2.7) holds on $\mathrm{Int}\, M \setminus \{x_0\}$. Choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of $T_{x_0}M$ with $e_{u,n} = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0_{x_0}$ and $Y'_{u,i}(0) = e_{u,i}$. By Lemma 2.9, for all i we have $Y_{u,i} = F_{\kappa,u}\, E_{u,i}$ on $[0, l]$, where $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$; moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_u$ is constant on $[0, l]$. Since the equalities in (6.2) hold, we have $s_{f,u}(l) = C_{\kappa,\lambda}$ and $F_{\kappa,u}(l) > 0$; in particular, we have no conjugate point of x_0 along γ_u. Define a diffeomorphism $\Phi : [0, l] \times U_{x_0}M \to M$ by $\Phi(t, u) := \gamma_u(t)$. By the rigidity of Jacobi fields, Φ is a Riemannian isometry with boundary from $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$ to M. Therefore, M is isometric to $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$.

Assume $N \in (-\infty, 1)$. In this case, for some constant $\delta \in \mathbb{R}$ we have $f = (n-1)\delta$ on M; in particular, for all $t \in [0, l]$,

$$F_{\kappa,u}(t) = e^{2\delta}\, s_\kappa(e^{-2\delta} t) = s_{\kappa e^{-4\delta}}(t), \quad l = e^{2\delta} C_{\kappa,\lambda} = C_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}.$$

It follows that $[0, l] \times_{F_\kappa} \mathbb{S}^{n-1}$ can be written as $B^n_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$. We complete the proof of Theorem 1.2. □

Furthermore, we prove Theorem 1.3.

Proof of Theorem 1.3. Let κ and λ satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. From Proposition 6.3 we derive (1.12).

Take $x_0 \in M$ with $\rho_{\partial M,\delta}(x_0) = C_{\kappa,\lambda}$. Put $l := \rho_{\partial M}(x_0)$ and

$$\Omega := \{x \in \operatorname{Int} M \setminus \{x_0\} \mid \rho_{\partial M}(x) + \rho_{x_0}(x) = l\}.$$

The set Ω is a non-empty closed subset of $\operatorname{Int} M \setminus \{x_0\}$.

We show that Ω is open in $\operatorname{Int} M \setminus \{x_0\}$. For each $x \in \Omega$, there exists an open neighborhood U of x such that ρ_{x_0} and $\rho_{\partial M}$ are smooth on U, and for every $y \in U$ there exists a unique minimal geodesic in M from x_0 to y that lies in $\operatorname{Int} M$. Let $\rho_{x_0,\delta} : M \to \mathbb{R}$ be a function defined by $\rho_{x_0,\delta} := e^{-2\delta}\rho_{x_0}$. By Lemma 2.8 and (5.3), for each $y \in U$

$$\begin{aligned} -\frac{\Delta_f(\rho_{\partial M} + \rho_{x_0})(y)}{(n-1)\, e^{\frac{-2f(y)}{n-1}}} &\leq \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M,\delta}(y)) + \frac{s'_{\kappa}}{s_{\kappa}}(\rho_{x_0,\delta}(y)) \\ &= \frac{s_{\kappa,\lambda}(\rho_{\partial M,\delta}(y) + \rho_{x_0,\delta}(y))}{s_{\kappa,\lambda}(\rho_{\partial M,\delta}(y))\, s_{\kappa}(\rho_{x_0,\delta}(y))} \leq 0. \end{aligned}$$

Lemma 2.5 implies $U \subset \Omega$, and hence Ω is open.

From the connectedness of $\operatorname{Int} M \setminus \{x_0\}$ we deduce $\Omega = \operatorname{Int} M \setminus \{x_0\}$, and hence $\rho_{\partial M} + \rho_{x_0} = l$ on M. This implies $M = B_l(x_0)$. For each $u \in U_{x_0}M$, we have $\tau_{x_0}(u) = l$, and γ_u is orthogonal to ∂M at l. The equality in (2.7) holds on $\operatorname{Int} M \setminus \{x_0\}$. Choose an orthonormal basis $\{e_{u,i}\}_{i=1}^n$ of $T_{x_0}M$ with $e_{u,n} = u$. Let $\{Y_{u,i}\}_{i=1}^{n-1}$ be the Jacobi fields along γ_u with initial conditions $Y_{u,i}(0) = 0_{x_0}$ and $Y'_{v,i}(0) = e_{u,i}$. For all i we have $Y_{u,i} = F_{\kappa,u}\, E_{u,i}$ on $[0,l]$, where $F_{\kappa,u}$ is the function defined as (1.9), and $\{E_{u,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_u with initial condition $E_{u,i}(0) = e_{u,i}$; moreover, $f \circ \gamma_u = (n-1)\delta$ on $[0,l]$ (see Remark 2.3 and Lemma 2.9). We see that $F_{\kappa,u} = s_{\kappa e^{-4\delta}}$ on $[0,l]$, and that $l = C_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$; in particular, we have no conjugate point of x_0 along γ_u. Define a diffeomorphism $\Phi : [0,l] \times U_{x_0}M \to M$ by $\Phi(t,u) := \gamma_u(t)$. By the rigidity of Jacobi fields, Φ is a Riemannian isometry with boundary from $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$ to M. It follows that M is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$. This proves Theorem 1.3. $\square$

Chapter 7

Splitting theorems

In this chapter, we prove various splitting theorems.

7.1 Busemann functions and asymptotes

A minimal geodesic $\gamma : [0, \infty) \to M$ is said to be a *ray*. For a ray $\gamma : [0, \infty) \to M$, the *Busemann function* $b_\gamma : M \to \mathbb{R}$ *of* γ is defined as

$$b_\gamma(x) := \lim_{t \to \infty} (t - d_M(x, \gamma(t))) .$$

For the Busemann functions, we show the following (see e.g., [45]):

Lemma 7.1. *Suppose that for some $z \in \partial M$ we have $\tau(z) = \infty$. Take $x \in \operatorname{Int} M$. If $b_{\gamma_z}(x) = \rho_{\partial M}(x)$, then $x \notin \operatorname{Cut} \partial M$. Moreover, for the unique foot point z_x on ∂M of x, we have $\tau(z_x) = \infty$.*

Proof. We prove by contradiction. Suppose $x \in \operatorname{Cut} \partial M$. Take $\epsilon > 0$ such that $B_\epsilon(x)$ is contained in $\operatorname{Int} M$, and a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, we take a minimal geodesic $\gamma_i : [0, l_i] \to M$ from x to $\gamma_z(t_i)$. It holds that

$$t_i - d_M(x, \gamma_z(t_i)) = -\epsilon + (t_i - d_M(\gamma_i(\epsilon), \gamma_z(t_i))). \tag{7.1}$$

Put $u_i := \gamma_i'(0) \in U_x M$. By taking a subsequence, for some $u \in U_x M$, we have $u_i \to u$ in $U_x M$. Letting $i \to \infty$ in (7.1), we see

$$b_{\gamma_z}(x) = -\epsilon + b_{\gamma_z}(\gamma_u(\epsilon)). \tag{7.2}$$

Since $\rho_{\partial M}$ is 1-Lipschitz, $b_{\gamma_z} \leq \rho_{\partial M}$ on M. By $b_{\gamma_z}(x) = \rho_{\partial M}(x)$ and (7.2),

$$\rho_{\partial M}(x) \leq -\epsilon + \rho_{\partial M}(\gamma_u(\epsilon)). \tag{7.3}$$

Note that (7.3) is the opposite inequality of the triangle inequality. Now, we take a foot point z_x on ∂M of x. Then (7.3) implies

$$d_M(z_x, \gamma_u(\epsilon)) \geq \rho_{\partial M}(\gamma_u(\epsilon)) \geq \rho_{\partial M}(x) + \epsilon = d_M(z_x, x) + d_M(x, \gamma_u(\epsilon));$$

in particular, $u = \gamma_{z_x}'(\tau(z_x))$. Furthermore, $\rho_{\partial M}(\gamma_{z_x}(\tau(z_x) + \epsilon)) = \tau(z_x) + \epsilon$. This contradicts the definition of τ. Hence $x \notin \operatorname{Cut} \partial M$, and z is the unique foot point.

Put $l := \rho_{\partial M}(x)$. For every sufficiently small $\epsilon > 0$, we see

$$b_{\gamma_z}(\gamma_z(l + \epsilon)) = \rho_{\partial M}(\gamma_z(l + \epsilon)),$$

and hence for all $t \geq l$ we have $b_{\gamma_z}(\gamma_z(t)) = \rho_{\partial M}(\gamma_z(t))$. This proves $\tau(z) = \infty$. $\square$

Let $\gamma : [0,\infty) \to M$ be a ray. Choose a point $x \in \operatorname{Int} M$, and a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, we take a minimal geodesic $\gamma_i : [0, l_i] \to M$ from x to $\gamma(t_i)$. We see $l_i \to \infty$. Take a sequence $\{T_j\}$ with $T_j \to \infty$. Since M is proper, we can take a subsequence $\{\gamma_{1,i}\}$ of $\{\gamma_i\}$, and a minimal geodesic $\gamma_{x,1} : [0, T_1] \to M$ from x to $\gamma_{x,1}(T_1)$ such that $\gamma_{1,i}|_{[0,T_1]}$ uniformly converges to $\gamma_{x,1}$. In this manner, take a subsequence $\{\gamma_{2,i}\}$ of $\{\gamma_{1,i}\}$ and a minimal geodesic $\gamma_{x,2} : [0, T_2] \to M$ from x to $\gamma_{x,2}(T_2)$ such that $\gamma_{2,i}|_{[0,T_2]}$ uniformly converges to $\gamma_{x,2}$, where $\gamma_{x,2}|_{[0,T_1]} = \gamma_{x,1}$. By means of a diagonal argument, we obtain a subsequence $\{\gamma_k\}$ of $\{\gamma_i\}$ and a ray γ_x in M such that for every $t > 0$ we have $\gamma_k(t) \to \gamma_x(t)$ as $k \to \infty$. We call such a ray γ_x an *asymptote for γ from x*.

For asymptotes for rays, we have the following (see e.g., [45]):

Lemma 7.2. *Suppose that for some $z \in \partial M$ we have $\tau(z) = \infty$. For $l > 0$, put $x := \gamma_z(l)$. Then there exists $\epsilon > 0$ such that for all $y \in B_\epsilon(x)$, all asymptotes for the ray γ_z from y lie in* $\operatorname{Int} M$.

Proof. The proof is by contradiction. Suppose that there exists a sequence $\{x_i\}$ in $\operatorname{Int} M$ with $x_i \to x$ such that for each i, there exists an asymptote γ_i for γ_z from x_i such that γ_i does not lie in $\operatorname{Int} M$. By taking a subsequence of $\{\gamma_i\}$, we may assume that there exists a ray $\gamma_x : [0,\infty) \to M$ such that for every $t \geq 0$, we have $\gamma_i(t) \to \gamma_x(t)$ as $i \to \infty$.

Fix i. Since γ_i is an asymptote for γ_z from x_i, there exists a sequence $\{t_{i_k}\}$ with $t_{i_k} \to \infty$ as $k \to \infty$, and for every k there exists a minimal geodesic γ_{i_k} in M from x_i to $\gamma_z(t_{i_k})$ such that for every $t > 0$ we have $\gamma_{i_k}(t) \to \gamma_i(t)$ as $k \to \infty$. For a fixed $t > 0$, it holds that

$$t_{i_k} - d_M(x_i, \gamma_z(t_{i_k})) = -t + \left(t_{i_k} - d_M(\gamma_{i_k}(t), \gamma_z(t_{i_k}))\right).$$

Letting $k \to \infty$, we see $b_{\gamma_z}(x_i) = -t + b_{\gamma_z}(\gamma_i(t))$. By letting $i \to \infty$, we obtain

$$b_{\gamma_z}(x) = -t + b_{\gamma_z}(\gamma_x(t)). \tag{7.4}$$

Since $\rho_{\partial M}$ is 1-Lipschitz, $b_{\gamma_z} \leq \rho_{\partial M}$ on M. By (7.4) and $b_{\gamma_z}(x) = \rho_{\partial M}(x)$,

$$\begin{aligned} d_M(\gamma_x(t), z) &\geq \rho_{\partial M}(\gamma_x(t)) \geq b_{\gamma_z}(\gamma_x(t)) \\ &= t + \rho_{\partial M}(x) = d_M(\gamma_x(t), x) + d_M(x, z). \end{aligned}$$

The equality in the triangle inequality holds. Hence $\gamma_x|_{[0,\infty)}$ coincides with $\gamma_z|_{[l,\infty)}$.

Now, we put

$$t_i := \sup\{t > 0 \mid \gamma_i([0,t)) \subset \operatorname{Int} M\}$$

and $z_i := \gamma_i(t_i) \in \partial M$. From $\rho_{\partial M}(z_i) = 0$ we deduce

$$b_{\gamma_z}(x_i) = -t_i + b_{\gamma_z}(z_i) \leq -t_i.$$

From $b_{\gamma_z}(x_i) \to l$, it follows that $\{t_i\}$ does not diverge. We may assume that for some $z_0 \in \partial M$, the sequence $\{z_i\}$ converges to z_0. Then γ_z passes through z_0. This contradicts that $\gamma_z|_{(0,\infty)}$ lies in $\operatorname{Int} M$. We conclude the lemma. $\square$

7.2 Main splitting theorems

To prove splitting theorems, we show the following (see e.g., [45]):

Lemma 7.3. *If there exists a connected component ∂M_0 of ∂M such that $\tau = \infty$ on ∂M_0, then ∂M is connected and* $\operatorname{Cut} \partial M = \emptyset$.

Proof. We put

$$TD_{\partial M_0} := \bigcup_{z \in \partial M_0} \{t\, u_x \mid t > 0\}.$$

By Lemma 3.7, $\exp^{\perp}|_{TD_{\partial M_0}}$ is a diffeomorphism onto its image. It follows that $\exp^{\perp}(TD_{\partial M_0})$ is open and closed in $\operatorname{Int} M$. Since $\operatorname{Int} M$ is connected, $\exp^{\perp}(TD_{\partial M_0})$ coincides with $\operatorname{Int} M$. This proves the lemma. $\square$

We prove Theorem 1.4.

Proof of Theorem 1.4. Let us assume that $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n, \infty)$, suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$.

Suppose that $z_0 \in \partial M$ satisfies $\tau(z_0) = \infty$. For the connected component ∂M_0 of ∂M containing z_0, we put

$$\Omega := \{z \in \partial M_0 \mid \tau(z) = \infty\}.$$

By the continuity of τ, the set Ω is closed in ∂M_0.

We prove that Ω is open in ∂M_0. Fix $z_1 \in \Omega$. Take $l > 0$, and put $x_0 := \gamma_{z_1}(l)$. There exists an open neighborhood U of x_0 contained in $\operatorname{Int} M \setminus \operatorname{Cut} \partial M$. Taking U smaller, we may assume that for each $x \in U$ the unique foot point on ∂M of x belongs to ∂M_0. By Lemma 7.2, there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(x_0)$, all asymptotes for γ_{z_1} from x lie in $\operatorname{Int} M$. We may assume $U \subset B_\epsilon(x_0)$. Fix $x_1 \in U$, and take an asymptote $\gamma_{x_1} : [0, \infty) \to M$ for γ_{z_1} from x_1. For $t > 0$, define a function $b_{\gamma_{z_1},t} : M \to \mathbb{R}$ by

$$b_{\gamma_{z_1},t}(x) := b_{\gamma_{z_1}}(x_1) + t - d_M(x, \gamma_{x_1}(t)).$$

We see that $b_{\gamma_{z_1},t} - \rho_{\partial M}$ is a support function of $b_{\gamma_{z_1}} - \rho_{\partial M}$ at x_1. Since γ_{x_1} lie in $\operatorname{Int} M$, for every $t > 0$ the function $b_{\gamma_{z_1},t}$ is smooth on a neighborhood of x_1. By Lemma 2.6, we have

$$\Delta_f b_{\gamma_{z_1},t}(x_1) \leq -H_{N,\kappa}(t),$$

where $H_{N,\kappa}$ is the function defined as (2.5). Note that $H_{N,\kappa}(t) \to -(N-1)\sqrt{|\kappa|}$ as $t \to \infty$. Furthermore, $\rho_{\partial M}$ is smooth on U, and by (4.10) we have

$$\Delta_f \rho_{\partial M} \geq (N-1)\sqrt{|\kappa|}$$

on U. Hence $b_{\gamma_{z_1}} - \rho_{\partial M}$ is f-subharmonic on U. The function $b_{\gamma_{z_1}} - \rho_{\partial M}$ takes the maximal value 0 at x_1. From Lemma 2.5 we derive $b_{\gamma_{z_1}} = \rho_{\partial M}$ on U. By Lemma 7.1, the set Ω is open in ∂M_0.

Since ∂M_0 is connected, we have $\Omega = \partial M_0$. By Lemma 7.3, ∂M is connected and $\operatorname{Cut} \partial M = \emptyset$. The equality in (4.10) holds on $\operatorname{Int} M$. For each $z \in \partial M$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. By Lemma

4.12, for all i we see $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0,\infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0,\infty)$. Define a diffeomorphism $\Phi : [0,\infty) \times \partial M \to M$ by $\Phi(t,z) := \gamma_z(t)$. By the rigidity of Jacobi fields, Φ is a Riemannian isometry with boundary from $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$ to M. We complete the proof of Theorem 1.4. $\Box$

We next prove Theorem 1.5.

Proof of Theorem 1.5. Let f be bounded from above. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$.

Let $z_0 \in \partial M$ satisfy $\tau(z_0) = \infty$. Let ∂M_0 be the connected component of ∂M containing z_0. Put

$$\Omega := \{z \in \partial M_0 \mid \tau(z) = \infty\}.$$

We see that Ω is closed in ∂M_0.

We show that Ω is open in ∂M_0. Fix $z_1 \in \Omega$. Take $l > 0$, and put $x_0 := \gamma_{z_1}(l)$. There exists an open neighborhood U of x_0 contained in $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. Taking U smaller, we may assume that for each $x \in U$ the unique foot point on ∂M of x belongs to ∂M_0. By Lemma 7.2, there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(x_0)$, all asymptotes for γ_{z_1} from x lie in $\mathrm{Int}\, M$. We may assume $U \subset B_\epsilon(x_0)$. For a fixed $x_1 \in U$, take an asymptote $\gamma_{x_1} : [0,\infty) \to M$ for γ_{z_1} from x_1. For $t > 0$, define a function $b_{\gamma_{z_1},t} : M \to \mathbb{R}$ as

$$b_{\gamma_{z_1},t}(x) := b_{\gamma_{z_1}}(x_1) + t - d_M(x, \gamma_{x_1}(t)).$$

We see that $b_{\gamma_{z_1},t} - \rho_{\partial M}$ is a support function of $b_{\gamma_{z_1}} - \rho_{\partial M}$ at x_1, and $b_{\gamma_{z_1},t}$ is smooth on a neighborhood of x_1. From Lemma 2.8 we deduce

$$\Delta_f b_{\gamma_{z_1},t}(x_1) \leq -H_{n,\kappa}(e^{\frac{-2\sup f}{n-1}} t)\, e^{\frac{-2f(x_1)}{n-1}}.$$

Note that $H_{n,\kappa}(s) \to -(n-1)\sqrt{|\kappa|}$ as $s \to \infty$. Using (4.16), we have

$$\Delta_f \rho_{\partial M} \geq (n-1)\sqrt{|\kappa|} e^{\frac{-2f}{n-1}}$$

on U. It follows that $b_{\gamma_{z_1}} - \rho_{\partial M}$ is f-subharmonic on U. Since $b_{\gamma_{z_1}} - \rho_{\partial M}$ takes the maximal value 0 at x_1, Lemma 2.5 implies $b_{\gamma_{z_1}} = \rho_{\partial M}$ on U. Lemma 7.1 tells us that Ω is open in ∂M_0.

By $\Omega = \partial M_0$ and Lemma 7.3, ∂M is connected and $\mathrm{Cut}\, \partial M = \emptyset$. Furthermore, the equality in (4.16) holds on $\mathrm{Int}\, M$. For each $z \in \partial M$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. By Lemma 4.12, for all i we have $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0,\infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0,\infty)$. Define a diffeomorphism $\Phi : [0,\infty) \times \partial M \to M$ by $\Phi(t,z) := \gamma_z(t)$. The rigidity of Jacobi fields implies that Φ is a Riemannian isometry with boundary from $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$ to M. We complete the proof of Theorem 1.5. $\Box$

From Proposition 2.10 we derive the following corollary of Theorem 1.5:

Corollary 7.4 ([47])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function bounded from above. Suppose $\mathrm{Ric}^1_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Suppose that for some $z_0 \in \partial M$ we have $\tau(z_0) = \infty$. Then exist a function $f_0 : [0,\infty) \to \mathbb{R}$, and a Riemannian metric h_0 on ∂M such that M is isometric to a warped product $\big([0,\infty)\times\partial M, dt^2 + e^{\frac{2f_0(t)}{n-1}} h_0\big)$.*

Lemma 3.6 implies the following corollary of Theorem 1.4:

Corollary 7.5 ([46])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. If M is non-compact and ∂M is compact, then M is isometric to $[0,\infty) \times_{\kappa,\lambda} \partial M$, and for all $z \in \partial M$ and $t \in [0,\infty)$ we have $(f \circ \gamma_z)(t) = f(z) + (N-n)\lambda t$.*

We also have the following corollary of Theorem 1.5:

Corollary 7.6 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function bounded from above. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. If M is non-compact and ∂M is compact, then M is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0,\infty)$.*

7.3 Weighted Ricci curvature on the boundary

The following lemma seems to be well-known, especially in a submanifold setting (see e.g., Proposition 9.36 in [4], and Lemma 5.4 in [45]).

Lemma 7.7. *Take $z \in \partial M$, and take a unit vector u in $T_z\partial M$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$ with $e_{z,1} = u$. Then we have*

$$\mathrm{Ric}_h(u) = \mathrm{Ric}_g(u) - K_g(u_z, u) + \operatorname{trace} A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{z,i})\|^2,$$

where h is the induced Riemannian metric on ∂M, and $K_g(u_z, u)$ is the sectional curvature at z in (M, g) determined by u_z and u.

By using Lemma 7.7, we show the following:

Lemma 7.8 ([47])**.** *Take $z \in \partial M$, and take a unit vector u in $T_z\partial M$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$ with $e_{z,1} = u$. Then for all $N \in (-\infty,\infty)$*

$$\begin{aligned}\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}^N_f(u) &+ g((\nabla f)_x, u_z)\, g(S(u,u), u_z) \\ &- K_g(u_z, u) + \operatorname{trace} A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{z,i})\|^2.\end{aligned} \tag{7.5}$$

Proof. We first assume $N \in (-\infty,\infty) \setminus \{n\}$. We see

$$\begin{aligned}&h((\nabla(f|_{\partial M}))_z, u) = g((\nabla f)_z, u),\\ &\mathrm{Hess}(f|_{\partial M})(u,u) = \mathrm{Hess}\, f(u,u) + g\left((\nabla f)_z, u_z\right)\, g\left(S(u,u), u_z\right).\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) &= \mathrm{Ric}_h(u) + \mathrm{Hess}(f|_{\partial M})(u,u) - \frac{h((\nabla (f|_{\partial M}))_z, u)^2}{(N-1)-(n-1)} \\
&= \mathrm{Ric}_h(u) + \mathrm{Hess}\, f(u,u) + g((\nabla f)_z, u_z)\, g(S(u,u), u_z) - \frac{g((\nabla f)_z, u)^2}{N-n}.
\end{aligned}$$

By Lemma 7.7, we have (7.5).

We next assume $N = n$. If f is constant, then $\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}_h(u)$ and $\mathrm{Ric}^N_f(u) = \mathrm{Ric}_g(u)$, and hence Lemma 7.7 implies (7.5). If f is not constant, then both the left hand side of (7.5) and the right hand side are equal to $-\infty$. This proves the lemma. □

From Lemma 7.8 we derive the following:

Lemma 7.9 ([48])**.** *Take $z \in \partial M$, and take a unit vector u in $T_z\partial M$. If M is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$, then for all $N \in (-\infty, \infty)$ we have*

$$\begin{aligned}
\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}^N_f(u) &+ (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} - \kappa\, e^{\frac{-4f(z)}{n-1}} \\
&- \lambda\, g((\nabla f)_z, u_z)\, e^{\frac{-2f(z)}{n-1}} + \frac{\mathrm{Hess}\, f(u_z, u_z)}{n-1}.
\end{aligned}$$

Proof. Let us choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$ with $e_{z,1} = u$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. Since M is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$, there exists a Riemannian isometry with boundary from M to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$. Then for all i we have $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. For all i

$$A_{u_z} e_{z,i} = -Y'_{z,i}(0) = -\left(\frac{g((\nabla f)_z, u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}} \right) e_{z,i}. \tag{7.6}$$

By (7.6), for all $i \neq 1$ we have $S(u, e_{z,i}) = 0_z$, and we also have

$$S(u,u) = -\left(\frac{g((\nabla f)_z, u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}} \right) u_z, \tag{7.7}$$

$$\mathrm{trace}\, A_{S(u,u)} = (n-1) \left(\frac{g((\nabla f)_z, u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}} \right)^2. \tag{7.8}$$

The sectional curvature $K_g(u_z, u)$ at z in (M, g) determined by u_z and u satisfies

$$\begin{aligned}
K_g(u_z, u) &= -g(Y''_{z,1}(0), u) \qquad (7.9)\\
&= -\left(\frac{\mathrm{Hess}\, f(u_z,u_z)}{n-1} + \left(\frac{g((\nabla f)_z, u_z)}{n-1} \right)^2 - \kappa e^{\frac{-4f(z)}{n-1}} \right).
\end{aligned}$$

Combining Lemma 7.8, (7.7), (7.8) and (7.9), we obtain

$$\begin{aligned}
\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}^N_f(u) &- \frac{g((\nabla f)_z, u_z)^2}{n-1} + \lambda\, g((\nabla f)_z, u_z)\, e^{\frac{-2f(z)}{n-1}} \\
&+ \frac{\mathrm{Hess}\, f(u_z,u_z)}{n-1} + \left(\frac{g((\nabla f)_z, u_z)}{n-1} \right)^2 - \kappa\, e^{\frac{-4f(z)}{n-1}} \\
&+ (n-1)\left(\frac{g((\nabla f)_z, u_z)}{n-1} - \lambda e^{\frac{-2f(z)}{n-1}} \right)^2 - \left(\frac{g((\nabla f)_z, u_z)}{n-1} \right)^2,
\end{aligned}$$

and hence

$$\mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) = \mathrm{Ric}^{N}_{f}(u) + (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} - \kappa\, e^{\frac{-4f(z)}{n-1}} \\ - \lambda\, g((\nabla f)_z, u_z)\, e^{\frac{-2f(z)}{n-1}} + \frac{\mathrm{Hess}\, f(u_z, u_z)}{n-1}.$$

We arrive at the desired equality. □

7.4 Multi-splitting

Let M_0 be a connected complete Riemannian manifold (without boundary). A minimal geodesic $\gamma : \mathbb{R} \to M_0$ is said to be a *line*.

Wylie [54] has proved the following splitting theorem of Cheeger-Gromoll type (see Theorem 1.2 and Corollary 1.3 in [54]):

Theorem 7.10 ([54]). *Let M_0 be a connected complete Riemannian manifold, and let $f_0 : M_0 \to \mathbb{R}$ be a smooth function bounded from above. For $N \in (-\infty, 1]$ we suppose $\mathrm{Ric}^N_{f_0, M_0} \geq 0$. If M_0 contains a line, then there exists a connected complete Riemannian manifold $\widetilde{M}_0$ such that M_0 is isometric to a warped product space over $\mathbb{R} \times \widetilde{M}_0$; moreover, if $N \in (-\infty, 1)$, then M_0 is isometric to $\mathbb{R} \times \widetilde{M}_0$.*

From Theorem 7.10 we derive the following corollary of Theorem 1.5:

Corollary 7.11 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function bounded from above. Let $\kappa \leq 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty, 1)$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Suppose that for some $z_0 \in \partial M$ we have $\tau(z_0) = \infty$. Then there exist an integer $k \in \{0, \dots, n-1\}$ and an $(n-1-k)$-dimensional, connected complete Riemannian manifold $\widetilde{\partial M}$ containing no line such that ∂M is isometric to $\mathbb{R}^k \times \widetilde{\partial M}$. In particular, M is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \left(\mathbb{R}^k \times \widetilde{\partial M}\right)$.*

Proof. By Theorem 1.5, M is isometric to $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$, and for each $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0, \infty)$. In this case, $g((\nabla f)_z, u_z) = 0$ and $\mathrm{Hess}\, f(u_z, u_z) = 0$. From Lemma 7.9, and from $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, for every unit vector u in $T_z \partial M$ we derive

$$\begin{aligned} \mathrm{Ric}^{N-1}_{f|_{\partial M}}(u) &= \mathrm{Ric}^{N}_{f}(u) + (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} - \kappa e^{\frac{-4f(z)}{n-1}} \\ &\geq \mathrm{Ric}^{N}_{f}(u) + (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} \\ &\geq (n-1)\kappa\, e^{\frac{-4f(z)}{n-1}} + (n-1)\lambda^2 e^{\frac{-4f(z)}{n-1}} = 0. \end{aligned}$$

We obtain $\mathrm{Ric}^{N-1}_{f|_{\partial M}, \partial M} \geq 0$. We now see that $N-1$ is smaller than 1. Furthermore, the function $f|_{\partial M}$ is bounded from above. Applying Theorem 7.10 to ∂M inductively, we conclude the corollary. □

7.5 Variants of splitting theorems

For manifolds with boundary whose boundaries are disconnected, several rigidity theorems have been obtained by Kasue [24] (and Croke and Kleiner [12], Ichida

[22]) under a lower Ricci curvature bound. We give generalizations of the rigidity theorems in our weighted setting.

For $A_1, A_2 \subset M$, we put $d_M(A_1, A_2) := \inf_{x_1 \in A_1, x_2 \in A_2} d_M(x_1, x_2)$.

The following has been proved in [24] (see Lemma 1.6 in [24]):

Lemma 7.12 ([24]). *Suppose that ∂M is disconnected. For the connected components $\{\partial M_i\}_{i=1,2,\dots}$ of ∂M, let ∂M_1 be compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Then there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. Furthermore, for every $i = 1, 2$ there exists $z_i \in \partial M_i$ such that $d_M(z_1, z_2) = D$. The minimal geodesic $\gamma : [0, D] \to M$ from z_1 to z_2 is orthogonal to ∂M both at z_1 and at z_2, and the restriction $\gamma|_{(0,D)}$ lies in $\operatorname{Int} M$.*

Wylie [54] has proved the following (see Theorem 5.1 in [54]):

Theorem 7.13 ([54]). *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. For the connected components $\{\partial M_i\}_{i=1,2,\dots}$ of ∂M, let ∂M_1 be compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. Then M is isometric to $[0, D] \times_{F_{0,0}} \partial M_1$, and for all $z \in \partial M_1$ and $t \in [0, D]$ we have $\operatorname{Ric}^N_f(\gamma'_z(t)) = 0$.*

For $\kappa > 0$ and $\lambda < 0$, put $D_{\kappa,\lambda} := \inf \{ t > 0 \mid s'_{\kappa,\lambda}(t) = 0 \}$.

Using Theorem 7.13, we conclude the following rigidity theorem:

Theorem 7.14 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. For the connected components $\{\partial M_i\}_{i=1,2,\dots}$ of ∂M, let ∂M_1 be compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Let $\kappa > 0$. For $N \in [n, \infty)$, suppose $\operatorname{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then $\lambda < 0$ and $D \leq 2D_{\kappa,\lambda}$. If $D = 2D_{\kappa,\lambda}$, then M is isometric to $[0, D] \times_{\kappa,\lambda} \partial M_1$, and for every $z \in \partial M_1$ we have $f \circ \gamma_z = f(z) - (N-n) \log s_{\kappa,\lambda}$ on $[0, D]$.*

Proof. By the monotonicity of Ric^N_f with respect to N, we see $\operatorname{Ric}^1_{f,M} \geq (N-1)\kappa$. If $\lambda \geq 0$, then Theorem 7.13 implies that M is isometric to $[0, D] \times_{F_{0,0}} \partial M_1$, and for all $z \in \partial M_1$ and $t \in [0, D]$ we have $\operatorname{Ric}^1_f(\gamma'_z(t)) = 0$. This contradicts $\kappa > 0$, and hence $\lambda < 0$.

We prove that if $D \geq D_{\kappa,\lambda}$, then M is isometric to $[0, D_{\kappa,\lambda}] \times_{\kappa,\lambda} \partial M_1$. Suppose $D \geq D_{\kappa,\lambda}$. Lemma 7.12 implies that there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1, 2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be a function defined by $\rho_{\partial M_i}(x) := d_M(x, \partial M_i)$. We put

$$\Omega := \{x \in \operatorname{Int} M \mid \rho_{\partial M_1}(x) + \rho_{\partial M_2}(x) = D\}.$$

The set Ω is a non-empty closed subset of $\operatorname{Int} M$.

We show that Ω is open in $\operatorname{Int} M$. Fix $x \in \Omega$. For each $i = 1, 2$, we take a foot point $z_{x,i} \in \partial M_i$ on ∂M_i of x such that $d_M(x, z_{x,i}) = \rho_{\partial M_i}(x)$. From the triangle inequality we derive $d_M(z_{x,1}, z_{x,2}) = D$. The minimal geodesic $\gamma : [0, D] \to M$ from $z_{x,1}$ to $z_{x,2}$ is orthogonal to ∂M at $z_{x,1}$ and at $z_{x,2}$. Furthermore, $\gamma|_{(0,D)}$ lies in $\operatorname{Int} M$ and passes through x. There exists an open neighborhood U of x such that

$\rho_{\partial M_i}$ is smooth on U. By using (4.10), for all $y \in U$, we see

$$-\frac{\Delta_f\ (\rho_{\partial M_1}+\rho_{\partial M_2})\,(y)}{N-1} \le \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M_1}(y)) + \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M_2}(y)) \tag{7.10}$$
$$= \frac{s'_{\kappa,\lambda}(\rho_{\partial M_1}(y)+\rho_{\partial M_2}(y)) - \lambda s_{\kappa,\lambda}(\rho_{\partial M_1}(y)+\rho_{\partial M_2}(y))}{s_{\kappa,\lambda}(\rho_{\partial M_1}(y))s_{\kappa,\lambda}(\rho_{\partial M_2}(y))}.$$

Since $\kappa > 0$, the function $s'_{\kappa,\lambda}/s_{\kappa,\lambda}$ is monotone decreasing on $(0, C_{\kappa,\lambda})$, and satisfies $s'_{\kappa,\lambda}(2D_{\kappa,\lambda})/s_{\kappa,\lambda}(2D_{\kappa,\lambda}) = \lambda$. By the triangle inequality, and by the assumption $D \ge D_{\kappa,\lambda}$, we have $\rho_{\partial M_1} + \rho_{\partial M_2} \ge 2D_{\kappa,\lambda}$ on U. The inequality (7.10) tells us that $-(\rho_{\partial M_1} + \rho_{\partial M_2})$ is f-subharmonic on U. By Lemma 2.5, Ω is open in $\operatorname{Int} M$.

The connectedness of $\operatorname{Int} M$ implies $\operatorname{Int} M = \Omega$. The equality in (4.10) holds. For each $z \in \partial M_1$, let us choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z}e_{z,i}$. By Lemma 4.10, for all i we see $Y_{z,i} = s_{\kappa,\lambda}E_{z,i}$ on $[0, D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = f(z) - (N-n)\log s_{\kappa,\lambda}$ on $[0, D]$. We see $D = 2D_{\kappa,\lambda}$. Define a diffeomorphism $\Phi : [0, D] \times \partial M_1 \to M$ by $\Phi(t, z) := \gamma_z(t)$. By the rigidity of Jacobi fields, Φ is a Riemannian isometry with boundary from $[0, D] \times_{\kappa,\lambda} \partial M_1$ to M. This completes the proof of the theorem. $\square$

Under the curvature bound (1.4), we obtain the following:

Theorem 7.15 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. For the connected components $\{\partial M_i\}_{i=1,2,\dots}$ of ∂M, let ∂M_1 be compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Let $\kappa > 0$. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}^N_{f,M} \ge (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \ge (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \le (n-1)\delta$ on M. Then $\lambda < 0$ and $D \le 2e^{2\delta}D_{\kappa,\lambda}$; moreover, if $D = 2e^{2\delta}D_{\kappa,\lambda}$, then M is isometric to $[0, D] \times_{F_{\kappa,\lambda}} \partial M_1$, and $f = (n-1)\delta$ on M.*

Proof. If we assume $\lambda \ge 0$, then Theorem 7.13 tells us that M is isometric to $[0, D] \times_{F_{0,0}} \partial M_1$, and for all $z \in \partial M_1$ and $t \in [0, D]$ we have $\operatorname{Ric}^N_f(\gamma'_z(t)) = 0$. This contradicts $\kappa > 0$, and hence $\lambda < 0$.

We prove that if $D \ge 2e^{2\delta}D_{\kappa,\lambda}$, then M is isometric to $[0, 2e^{2\delta}D_{\kappa,\lambda}] \times_{F_{\kappa,\lambda}} \partial M_1$, and $f = (n-1)\delta$ on M. We suppose $D \ge 2e^{2\delta}D_{\kappa,\lambda}$. By Lemma 7.12, there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1, 2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be a function defined as $\rho_{\partial M_i}(x) := d_M(x, \partial M_i)$. Put

$$\Omega := \{x \in \operatorname{Int} M \mid \rho_{\partial M_1}(x) + \rho_{\partial M_2}(x) = D\}$$

Note that Ω is a non-empty closed subset of $\operatorname{Int} M$.

We prove that Ω is open in $\operatorname{Int} M$. For a fixed $x \in \Omega$, and for each $i = 1, 2$, take a foot point $z_{x,i} \in \partial M_i$ on ∂M_i of x such that $d_M(x, z_{x,i}) = \rho_{\partial M_i}(x)$. We see $d_M(z_{x,1}, z_{x,2}) = D$. The minimal geodesic $\gamma : [0, D] \to M$ from $z_{x,1}$ to $z_{x,2}$ is orthogonal to ∂M at $z_{x,1}$ and at $z_{x,2}$, and $\gamma|_{(0,D)}$ lies in $\operatorname{Int} M$ and passes through x. There exists an open neighborhood U of x such that $\rho_{\partial M_i}$ is smooth on U. Using

(5.3), for all $y \in U$, we have

$$
\begin{aligned}
-\frac{\Delta_f\,(\rho_{\partial M_1}+\rho_{\partial M_2})\,(y)}{(n-1)e^{\frac{-2f(y)}{n-1}}} &\leq \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M_1,\delta}(y)) + \frac{s'_{\kappa,\lambda}}{s_{\kappa,\lambda}}(\rho_{\partial M_2,\delta}(y)) \qquad (7.11) \\
&= \frac{s'_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y)+\rho_{\partial M_2,\delta}(y)) - \lambda s_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y)+\rho_{\partial M_2,\delta}(y))}{s_{\kappa,\lambda}(\rho_{\partial M_1,\delta}(y))s_{\kappa,\lambda}(\rho_{\partial M_2,\delta}(y))},
\end{aligned}
$$

where $\rho_{\partial M_i,\delta} : M \to \mathbb{R}$ denotes a function defined as $\rho_{\partial M_i,\delta} := e^{-2\delta}\rho_{\partial M_i}$. From the triangle inequality and $D \geq 2e^{2\delta}D_{\kappa,\lambda}$ we deduce $\rho_{\partial M_1,\delta}+\rho_{\partial M_2,\delta} \geq 2D_{\kappa,\lambda}$ on U. From (7.11), it follows that $-(\rho_{\partial M_1}+\rho_{\partial M_2})$ is f-subharmonic on U. Lemma 2.5 implies that Ω is open in $\operatorname{Int} M$.

Since $\operatorname{Int} M$ is connected, we have $\operatorname{Int} M = \Omega$. Furthermore, the equality in (5.3) holds. For each $z \in \partial M_1$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z}e_{z,i}$. For all i we see $Y_{z,i} = F_{\kappa,\lambda,z}E_{z,i}$ on $[0,D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = (n-1)\delta$ on $[0,D]$ (see Remark 5.3). We see $D = 2e^{2\delta}D_{\kappa,\lambda}$. Define a diffeomorphism $\Phi : [0,D] \times \partial M_1 \to M$ by $\Phi(t,z) := \gamma_z(t)$. From the rigidity of Jacobi fields, we conclude that Φ is a Riemannian isometry with boundary from $[0,D] \times_{F_{\kappa,\lambda}} \partial M_1$ to M. This proves the theorem. □

Furthermore, we prove the following:

Theorem 7.16 ([47]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is disconnected. For the connected components $\{\partial M_i\}_{i=1,2,\dots}$ of ∂M, let ∂M_1 be compact. Put $D := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. Then M is isometric to $[0,D] \times_{F_{0,0}} \partial M_1$; moreover, if $N \in (-\infty,1)$, then for every $z \in \partial M_1$ the function $f \circ \gamma_z$ is constant on $[0,D]$.*

Proof. By Lemma 7.12, there exists a connected component ∂M_2 of ∂M such that $d_M(\partial M_1, \partial M_2) = D$. For each $i = 1,2$, let $\rho_{\partial M_i} : M \to \mathbb{R}$ be the distance function from ∂M_i defined as $\rho_{\partial M_i}(x) := d_M(x, \partial M_i)$. Put

$$\Omega := \{x \in \operatorname{Int} M \mid \rho_{\partial M_1}(x) + \rho_{\partial M_2}(x) = D\}.$$

We show the openness of Ω in $\operatorname{Int} M$. For a fixed point $x \in \Omega$, we see that there exists an open neighborhood U of x such that $\rho_{\partial M_i}$ is smooth on U. By using Lemma 4.6, we have $\Delta_f\, \rho_{\partial M_i} \geq 0$ on U; in particular, $-(\rho_{\partial M_1}+\rho_{\partial M_2})$ is f-subharmonic on U. From Lemma 2.5, it follows that Ω is open in $\operatorname{Int} M$.

We have $\operatorname{Int} M = \Omega$, and the equality in (4.22) holds. For each $z \in \partial M_1$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z}e_{z,i}$. By using Lemma 4.14, for all i we have $Y_{z,i} = F_{0,0,z}E_{z,i}$ on $[0,D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty,1)$, then for every $z \in \partial M_1$ the function $f \circ \gamma_z$ is constant on $[0,D]$. Define a map $\Phi : [0,D] \times \partial M_1 \to M$ by $\Phi(t,z) := \gamma_z(t)$. By the rigidity of Jacobi fields, we see that Φ is a Riemannian isometry with boundary from $[0,D] \times_{F_{0,0}} \partial M_1$ to M. This completes the proof. □

Chapter 8

Volume comparisons

In the present chapter, we prove several volume comparison results for the neighborhoods of the boundaries, and conclude volume growth rigidity theorems.

8.1 Volume element comparisons

Let $z \in \partial M$. For $t \in (0, \tau(z))$, we denote by $\theta(t, z)$ the absolute value of the Jacobian of $\exp^{\perp}$ at $(z, tu_z) \in T^{\perp}\partial M$. We see that $\theta(t, z)$ is equal to the volume element of the t-level surface of $\rho_{\partial M}$ at $\gamma_z(t)$, and satisfies

$$\Delta\, \rho_{\partial M}(\gamma_z(t)) = -\frac{\theta'(t, z)}{\theta(t, z)}. \tag{8.1}$$

We put

$$\theta_f(t, z) := e^{-f(\gamma_z(t))}\, \theta(t, z). \tag{8.2}$$

By using (8.1), we see

$$\Delta_f\, \rho_{\partial M}(\gamma_z(t)) = -\frac{\theta'_f(t, z)}{\theta_f(t, z)}. \tag{8.3}$$

By Lemma 4.4, we have the following volume element comparison result:

Lemma 8.1 ([46]). *Take $z \in \partial M$. For $N \in [n, \infty)$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq (N-1)\kappa$, and suppose $H_{f,z} \geq (N-1)\lambda$. Then for all $t_1, t_2 \in [0, \tau(z))$ with $t_1 \leq t_2$ we have*

$$\frac{\theta_f(t_2, z)}{\theta_f(t_1, z)} \leq \frac{s^{N-1}_{\kappa,\lambda}(t_2)}{s^{N-1}_{\kappa,\lambda}(t_1)},$$

where $\theta_f(t, z)$ is defined as (8.2). In particular, for all $t \in [0, \tau(z))$ we have

$$\theta_f(t, z) \leq e^{-f(z)}\, s^{N-1}_{\kappa,\lambda}(t). \tag{8.4}$$

Proof. By (4.10) and (8.3), for all $t \in (0, \tau(z))$

$$\frac{d}{dt} \log \frac{\theta_f(t, z)}{s^{N-1}_{\kappa,\lambda}(t)} = -\Delta_f \rho_{\partial M}(\gamma_z(t)) + H_{N,\kappa,\lambda}(t) \leq 0,$$

where $H_{N,\kappa,\lambda}$ is the function defined as (4.8). This implies the lemma. $\square$

Remark 8.1. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (8.4) holds. Then the equality in (8.4) holds on $[0, t_0]$; in particular, the equality in (4.10) holds on $[0, t_0]$ (see Lemma 4.10).

For $s \in (0, \tau_f(z))$, we put

$$\hat{\theta}_f(s, z) := \theta_f(t_{f,z}(s), z), \tag{8.5}$$

where $t_{f,z}$ denotes the inverse function of $s_{f,z}$.

From Lemma 4.5 we deduce the following comparison result:

Lemma 8.2 ([48])**.** *Take a point $z \in \partial M$. For $N \in (-\infty, 1]$, we suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (n-1)\kappa e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and we suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. Then for all $s_1, s_2 \in [0, \tau_f(z))$ with $s_1 \leq s_2$ we have*

$$\frac{\hat{\theta}_f(s_2, z)}{\hat{\theta}_f(s_1, z)} \leq \frac{s_{\kappa,\lambda}^{n-1}(s_2)}{s_{\kappa,\lambda}^{n-1}(s_1)},$$

where $\hat{\theta}_f(s, z)$ is defined as (8.5). In particular, for all $s \in [0, \tau_f(z))$ we have

$$\hat{\theta}_f(s, z) \leq e^{-f(z)}\, s_{\kappa,\lambda}^{n-1}(s). \tag{8.6}$$

Proof. From (4.15) and (8.3), for all $s \in (0, \tau_f(z))$ we deduce

$$\frac{d}{ds} \log \frac{\hat{\theta}_f(s, z)}{s_{\kappa,\lambda}^{n-1}(s)} = -\left(e^{\frac{2f}{n-1}}\, \Delta_f \rho_{\partial M}\right)(\gamma_z(t_{f,z}(s))) + H_{n,\kappa,\lambda}(s) \leq 0.$$

This leads us to the desired inequality. $\square$

Remark 8.2. Assume that for some $s_0 \in (0, \tau_f(z))$ the equality in (8.6) holds. Then the equality in (8.6) holds on $[0, s_0]$; in particular, the equality in (4.15) holds on $[0, s_0]$ (see Lemma 4.11).

For $\delta \in \mathbb{R}$, the function $\tau_\delta : \partial M \to (0, \infty]$ is defined as $\tau_\delta := e^{-2\delta}\tau$. For $s \in (0, \tau_\delta(z))$, we put

$$\hat{\theta}_{f,\delta}(s, z) := \theta_f(e^{2\delta}s, z). \tag{8.7}$$

If f is bounded from above, then Lemma 5.3 implies the following:

Lemma 8.3 ([48])**.** *Take a point $z \in \partial M$. Let us assume that κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}_f^N(\gamma_z'(t)) \geq (n-1)\kappa\, e^{\frac{-4f(\gamma_z(t))}{n-1}}$, and suppose $H_{f,z} \geq (n-1)\lambda e^{\frac{-2f(z)}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \circ \gamma_z \leq (n-1)\delta$ on $(0, \tau(z))$. Then for all $s_1, s_2 \in [0, \tau_\delta(z))$ with $s_1 \leq s_2$*

$$\frac{\hat{\theta}_{f,\delta}(s_2, z)}{\hat{\theta}_{f,\delta}(s_1, z)} \leq \frac{s_{\kappa,\lambda}^{n-1}(s_2)}{s_{\kappa,\lambda}^{n-1}(s_1)},$$

where $\hat{\theta}_{f,\delta}(s, z)$ is defined as (8.7). In particular, for all $s \in [0, \tau_\delta(z))$ we have

$$\hat{\theta}_{f,\delta}(s, z) \leq e^{-f(z)}\, s_{\kappa,\lambda}^{n-1}(s) \tag{8.8}$$

Proof. By (5.4) and (8.3), for all $s \in (0, \tau_\delta(z))$ we see

$$\frac{d}{ds} \log \frac{\hat{\theta}_{f,\delta}(s,z)}{s^{n-1}_{\kappa,\lambda}(s)} = -\Delta_f \rho_{\partial M} \left(\gamma_z(e^{2\delta} s) \right) e^{2\delta} + H_{n,\kappa,\lambda}(s) \leq 0.$$

This implies the desired inequality. □

Remark 8.3. Assume that for some $s_0 \in (0, \tau_\delta(z))$ the equality in (8.8) holds. Then the equality in (8.8) holds on $[0, s_0]$; in particular, the equality in (5.4) holds on $[0, e^{2\delta} s_0]$ (see Remark 5.4).

If the curvatures are bounded by constants, then we have the following:

Lemma 8.4. *Take $z \in \partial M$. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq \kappa$, and suppose $H_{f,z} \geq \lambda$. Then for all $t_1, t_2 \in [0, \tau(z))$ with $t_1 \leq t_2$ we have*

$$\theta_f(t_2, z) \leq \exp \left(-\int_{t_1}^{t_2} F^{-2}_{0,0,z}(a) \left(\kappa \int_0^a F^2_{0,0,z}(b)\, db + \lambda \right) da \right) \theta_f(t_1, z),$$

where $\theta_f(t,z)$ is defined as (8.2)*. In particular, for all $t \in [0, \tau(z))$ we have*

$$\theta_f(t, z) \leq \exp \left(-\int_0^t F^{-2}_{0,0,z}(a) \left(\kappa \int_0^a F^2_{0,0,z}(b)\, db + \lambda \right) da \right) e^{-f(z)}. \tag{8.9}$$

Proof. Using (4.21) and (8.3), for all $t \in (0, \tau(z))$ we have

$$\begin{aligned} &\frac{d}{dt} \log \frac{\theta_f(t,z)}{\exp\left(-\int_0^t F^{-2}_{0,0,z}(a) \left(\kappa \int_0^a F^2_{0,0,z}(b)\, db + \lambda \right) da \right)} \\ &\qquad = -\Delta_f \rho_{\partial M}(\gamma_z(t)) + F^{-2}_{0,0,z}(t) \left(\kappa \int_0^t F^2_{0,0,z}(a)\, da + \lambda \right) \leq 0. \end{aligned}$$

This proves the lemma. □

Remark 8.4. Assume that for some $t_0 \in (0, \tau(z))$ the equality in (8.9) holds. Then the equality in (8.9) holds on $[0, t_0]$; in particular, the equality in (4.21) holds on $[0, t_0]$ (see Lemma 4.13).

By Lemma 8.4, we see the following:

Lemma 8.5 ([47])**.** *Take $z \in \partial M$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose that for all $t \in (0, \tau(z))$ we have $\mathrm{Ric}^N_f(\gamma'_z(t)) \geq \kappa$, and suppose $H_{f,z} \geq \lambda$. Then for all $t_1, t_2 \in [0, \tau(z))$ with $t_1 \leq t_2$ we have*

$$\theta_f(t_2, z) \leq \theta_f(t_1, z),$$

where $\theta_f(t,z)$ is defined as (8.2)*. In particular, for all $t \in [0, \tau(z))$ we have*

$$\theta_f(t, z) \leq e^{-f(z)}. \tag{8.10}$$

Remark 8.5. If for some $t_0 \in (0, \tau(z))$ the equality in (8.10) holds, then the equality in (8.9) also holds (see Remark 8.4).

8.2 Integration formulas

By Lemma 3.1, we see the following (see e.g., [45]):

Lemma 8.6. *For all $r > 0$, we have*

$$B_r(\partial M) = \bigcup_{z \in \partial M} \{\gamma_z(t) \mid t \in [0, \min\{r, \tau(z)\}]\}.$$

Proof. We first show

$$B_r(\partial M) \subset \bigcup_{z \in \partial M} \{\gamma_z(t) \mid t \in [0, \min\{r, \tau(z)\}]\}. \tag{8.11}$$

Take $x \in B_r(\partial M)$, and a foot point z on ∂M of x. From Lemma 3.1 we deduce $x = \gamma_z(l)$, where $l = \rho_{\partial M}(x)$. We see $l \leq \tau(z)$. Since $x \in B_r(\partial M)$, we have $l \leq r$. This proves (8.11).

We show the opposite inclusion of (8.11). Take $z \in \partial M$ and $t \in [0, \min\{r, \tau(z)\}]$. From $t \leq \tau(z)$, it follows that

$$\rho_{\partial M}(\gamma_z(t)) = t \leq r;$$

in particular, we see the opposite. □

We define a function $\bar{\theta}_f : [0, \infty) \times \partial M \to \mathbb{R}$ by

$$\bar{\theta}_f(t, z) := \begin{cases} \theta_f(t, z) & \text{if } t < \tau(z), \\ 0 & \text{if } t \geq \tau(z), \end{cases} \tag{8.12}$$

where $\theta_f(t, z)$ is defined as (8.12).

For the proof of volume comparisons, we show the following (see e.g., [46]):

Lemma 8.7. *If ∂M is compact, then for all $r > 0$ we have*

$$m_f(B_r(\partial M)) = \int_{\partial M} \int_0^r \bar{\theta}_f(t, z)\, dt\, d\,\mathrm{vol}_h,$$

where $\bar{\theta}_f$ is the function defined as (8.12).

Proof. Since ∂M is compact, $B_r(\partial M)$ is also compact; in particular, $m_f(B_r(\partial M))$ is finite. Lemma 8.6 implies

$$B_r(\partial M) = \exp^{\perp}\left(\bigcup_{z \in \partial M} \{t u_z \mid t \in [0, \min\{r, \tau(z)\}]\} \right).$$

By Lemma 3.7, the restriction of the map $\exp^{\perp}$ to the set

$$\bigcup_{z \in \partial M} \{t u_z \mid t \in (0, \min\{r, \tau(z)\})\}$$

is a diffeomorphism onto its image. Proposition 3.9 tells us that $\mathrm{Cut}\,\partial M$ is a null set. Hence, by using the coarea formula (see e.g., Theorem 3.2.3 in [14]) and the Fubini theorem, we obtain

$$m_f(B_r(\partial M)) = \int_{\partial M} \int_0^{\min\{r, \tau(z)\}} \theta_f(t, z)\, dt\, d\,\mathrm{vol}_h = \int_{\partial M} \int_0^r \bar{\theta}_f(t, z)\, dt\, d\,\mathrm{vol}_h\,.$$

We arrive at the desired equation. □

For $r > 0$, we put

$$U_{r,f} := \{z \in \partial M \mid \tau_f(z) \le r\}, \quad \widehat{U}_{r,f} := \bigcup_{z \in U_{r,f}} \{\gamma_z(t) \mid t \in [0, \tau(z))\}$$
$$V_{r,f} := \{z \in \partial M \mid \tau_f(z) > r\}, \quad \widehat{V}_{r,f} := \bigcup_{z \in V_{r,f}} \{\gamma_z(t) \mid t \in [0, t_{f,z}(r)]\}.$$

We show the following:

Lemma 8.8 ([48])**.** *For all $r > 0$ we have*

$$B_{r,f}(\partial M) \setminus \mathrm{Cut}\, \partial M = \widehat{U}_{r,f} \sqcup \widehat{V}_{r,f}.$$

Proof. We first show

$$B_{r,f}(\partial M) \setminus \mathrm{Cut}\, \partial M \subset \widehat{U}_{r,f} \sqcup \widehat{V}_{r,f}. \tag{8.13}$$

Take $x \in B_{r,f}(\partial M) \setminus \mathrm{Cut}\, \partial M$, and a unique foot point z on ∂M of x. By Lemma 3.1, we have $x = \gamma_z(l)$, where $l := \rho_{\partial M}(x)$. If $z \in U_{r,f}$, then $x \in \widehat{U}_{r,f}$. If $z \in V_{r,f}$, then it holds that

$$s_{f,z}(l) = \rho_{\partial M,f}(x) \le r < \tau_f(z),$$

and hence $l \le t_{f,z}(r)$. This proves (8.13).

We next show the opposite of (8.13). For all $z \in U_{r,f}$ and $t \in [0, \tau(z))$, we see

$$\rho_{\partial M,f}(\gamma_z(t)) = s_{f,z}(t) < \tau_f(z) \le r.$$

It follows that $\widehat{U}_{r,f} \subset B_{r,f}(\partial M) \setminus \mathrm{Cut}\, \partial M$. Furthermore, for all $z \in V_{r,f}$ and $t \in [0, t_{f,z}(r)]$, we see $t_{f,z}(r) < \tau(z)$ and $s_{f,z}(t) \le r$; in particular,

$$\rho_{\partial M,f}(\gamma_z(t)) = s_{f,z}(t) \le r.$$

Hence we obtain $\widehat{V}_{r,f} \subset B_{r,f}(\partial M) \setminus \mathrm{Cut}\, \partial M$. We have the opposite of (8.13). This completes the proof. $\square$

We define a function $\check{\theta}_f : [0, \infty) \times \partial M \to \mathbb{R}$ by

$$\check{\theta}_f(s, z) := \begin{cases} \hat{\theta}_f(s, z) & \text{if } s < \tau_f(z), \\ 0 & \text{if } s \ge \tau_f(z), \end{cases} \tag{8.14}$$

where $\hat{\theta}_f(s, z)$ is defined as (8.5).

We also prove the following integration formula:

Lemma 8.9 ([48])**.** *Suppose that ∂M is compact. Then for all $r > 0$*

$$m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M)) = \int_{\partial M} \int_0^r \check{\theta}_f(s, z)\, ds\, d\,\mathrm{vol}_h,$$

where $\check{\theta}_f$ is the function defined as (8.14).

Proof. From Proposition 3.9 and Lemma 8.8 we derive

$$m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M)) = m_{\frac{n+1}{n-1}f}\left(\widehat{U}_{r,f}\right) + m_{\frac{n+1}{n-1}f}\left(\widehat{V}_{r,f}\right).$$

Put

$$C_U := \int_{U_{r,f}} \int_0^{\tau(z)} e^{\frac{-(n+1)f(\gamma_z(t))}{n-1}}\,\theta(t,z)\,dt\,d\,\mathrm{vol}_h,$$

$$C_V := \int_{V_{r,f}} \int_0^{t_{f,z}(r)} e^{\frac{-(n+1)f(\gamma_z(t))}{n-1}}\,\theta(t,z)\,dt\,d\,\mathrm{vol}_h\,.$$

From the coarea formula and the Fubini theorem we deduce

$$m_{\frac{n+1}{n-1}f}\left(\widehat{U}_{r,f}\right) = C_U, \quad m_{\frac{n+1}{n-1}f}\left(\widehat{V}_{r,f}\right) = C_V.$$

Furthermore, it holds that

$$\begin{aligned} C_U &= \int_{U_{r,f}} \int_0^{\tau(z)} \theta_f(t,z)\, s'_{f,z}(t)\,dt\,d\,\mathrm{vol}_h \\ &= \int_{U_{r,f}} \int_0^{\tau_f(z)} \hat{\theta}_f(s,z)\,ds\,d\,\mathrm{vol}_h = \int_{U_{r,f}} \int_0^{r} \check{\theta}_f(s,z)\,ds\,d\,\mathrm{vol}_h, \end{aligned}$$

and similarly,

$$\begin{aligned} C_V &= \int_{V_{r,f}} \int_0^{t_{f,z}(r)} \theta_f(t,z)\, s'_{f,z}(t)\,dt\,d\,\mathrm{vol}_h \\ &= \int_{V_{r,f}} \int_0^{r} \hat{\theta}_f(s,z)\,ds\,d\,\mathrm{vol}_h = \int_{V_{r,f}} \int_0^{r} \check{\theta}_f(s,z)\,ds\,d\,\mathrm{vol}_h\,. \end{aligned}$$

Hence we have

$$m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M)) = C_U + C_V = \int_{\partial M} \int_0^r \check{\theta}_f(s,z)\,ds\,d\,\mathrm{vol}_h\,.$$

We obtain the desired equation. □

For $\delta \in \mathbb{R}$ and $r > 0$, put $B_{r,\delta}(\partial M) := \{\, x \in M \mid \rho_{\partial M,\delta}(x) \leq r \,\}$. We define a function $\check{\theta}_{f,\delta} : [0,\infty) \times \partial M \to \mathbb{R}$ by

$$\check{\theta}_{f,\delta}(s,z) := \begin{cases} \hat{\theta}_{f,\delta}(s,z) & \text{if } s < \tau_\delta(z), \\ 0 & \text{if } s \geq \tau_\delta(z), \end{cases}$$

where $\hat{\theta}_{f,\delta}(s,z)$ is defined as (8.7).

We have the following integration formula:

Lemma 8.10 ([48])**.** *Suppose that ∂M is compact. Let $\delta \in \mathbb{R}$. Then for all $r > 0$*

$$m_f(B_{r,\delta}(\partial M)) = e^{2\delta} \int_{\partial M} \int_0^r \check{\theta}_{f,\delta}(s,z)\,ds\,d\,\mathrm{vol}_h\,.$$

Proof. We see $B_{r,\delta}(\partial M) = B_{e^{2\delta}r}(\partial M)$. Using Lemma 8.7, we have

$$\begin{aligned} m_f(B_{r,\delta}(\partial M)) &= \int_{\partial M} \int_0^{e^{2\delta}r} \bar{\theta}_f(t,z)\,dt\,d\,\mathrm{vol}_h \\ &= e^{2\delta} \int_{\partial M} \int_0^r \check{\theta}_{f,\delta}(s,z)\,ds\,d\,\mathrm{vol}_h\,. \end{aligned}$$

We conclude the lemma. □

8.3 Absolute volume comparisons

Bayle [3] has stated the following absolute volume comparison inequality of Heintze-Karcher type without proof (see Theorem E.2.2 in [3], and also [37]).

Lemma 8.11 ([3]). *Suppose that ∂M is compact. For $N \in [n, \infty)$, suppose that we have $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then for all $r > 0$*

$$m_f(B_r(\partial M)) \leq s_{N,\kappa,\lambda}(r)\, m_{f,\partial M}(\partial M),$$

where $s_{N,\kappa,\lambda}$ is the function defined as (1.17). *In particular, we have* (1.19).

Proof. By Lemma 8.1, for all $t \geq 0$ we see

$$\bar{\theta}_f(t,z) \leq e^{-f(z)}\, \bar{s}^{N-1}_{\kappa,\lambda}(t),$$

where $\bar{s}_{\kappa,\lambda}$ is the function defined as (1.17). Integrate the both sides over $[0, r]$ with respect to t, and over ∂M with respect to z. Lemma 8.7 implies the desired inequality. □

Under the curvature bound (1.4), we prove the following:

Lemma 8.12 ([48]). *Suppose that ∂M is compact. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Then for all $r > 0$*

$$m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M)) \leq s_{n,\kappa,\lambda}(r)\, m_{f,\partial M}(\partial M).$$

In particular, we have (1.22).

Proof. By Lemma 8.2, for all $s \geq 0$

$$\check{\theta}_f(s,z) \leq e^{-f(z)}\, \bar{s}^{n-1}_{\kappa,\lambda}(s),$$

Integrate the both sides over $[0, r]$ with respect to s, and over ∂M with respect to z. From Lemma 8.9, we conclude the lemma. □

In the case where f is bounded, we have the following:

Lemma 8.13 ([48]). *Suppose that ∂M is compact. Let us assume that κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then for all $r > 0$ we have*

$$m_f(B_{r,\delta}(\partial M)) \leq e^{2\delta} s_{n,\kappa,\lambda}(r)\, m_{f,\partial M}(\partial M).$$

In particular,

$$\limsup_{r\to\infty} \frac{m_f(B_{r,\delta}(\partial M))}{s_{n,\kappa,\lambda}(r)} \leq e^{2\delta}\, m_{f,\partial M}(\partial M).$$

Proof. Lemma 8.3 implies that for all $s \geq 0$

$$\check{\theta}_{f,\delta}(s,z) \leq e^{-f(z)}\, \bar{s}^{n-1}_{\kappa,\lambda}(s).$$

We integrate the both sides over $[0, r]$ with respect to s, and over ∂M with respect to z. From Lemma 8.10 we deduce the lemma. □

Furthermore, we show the following:

Lemma 8.14 ([47]). *Suppose that ∂M is compact. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. Then for all $r > 0$ we have*

$$m_f(B_r(\partial M)) \leq \int_{\partial M} \int_0^{\min\{r,\tau(z)\}} \exp\left(-\int_0^t F^{-2}_{0,0,z}(a)\left(\kappa \int_0^a F^2_{0,0,z}(b)\,db + \lambda\right)da\right) dt\,dm_{f,\partial M}.$$

In particular, if κ and λ satisfy the subharmonic-condition, then for all $r > 0$

$$m_f(B_r(\partial M)) \leq r\, m_{f,\partial M}(\partial M),$$

and hence

$$\limsup_{r\to\infty} \frac{m_f(B_r(\partial M))}{r} \leq m_{f,\partial M}(\partial M).$$

Proof. Define a function $\theta_{\kappa,\lambda} : [0,\infty) \times \partial M \to \mathbb{R}$ by

$$\theta_{\kappa,\lambda}(t,z) := \begin{cases} \exp\left(-\int_0^t F^{-2}_{0,0,z}(a)\left(\kappa \int_0^a F^2_{0,0,z}(b)\,db + \lambda\right)da\right) & \text{if } t < \tau(x), \\ 0 & \text{if } t \geq \tau(x). \end{cases}$$

By Lemma 8.4, for all $z \in \partial M$ and $t \geq 0$ we see

$$\bar{\theta}_f(t,z) \leq \theta_{\kappa,\lambda}(t,z)\, e^{-f(z)}.$$

Integrate the both sides of the inequality over $(0, r)$ with respect to t, and then do that over ∂M with respect to z. From Lemma 8.7 we deduce

$$m_f(B_r(\partial M)) \leq \int_{\partial M} \int_0^r \theta_{\kappa,\lambda}(t,z)\,dt\,dm_{f,\partial M}.$$

This implies the lemma. $\square$

Remark 8.6. Under a lower N-weighted Ricci curvature bound by constants, Morgan [39] has proved an inequality of Heintze-Karcher type in the case of $N = \infty$, and Milman [38] has done in the case of $N \in (-\infty, 1)$.

8.4 Relative volume comparisons

We prove the following relative volume comparison theorem:

Theorem 8.15 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. For $N \in [n, \infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Then for all $r, R > 0$ with $r \leq R$*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{s_{N,\kappa,\lambda}(R)}{s_{N,\kappa,\lambda}(r)}. \tag{8.15}$$

Proof. From Lemma 8.1, for all $t_1, t_2 \geq 0$ with $t_1 \leq t_2$ we deduce

$$\bar{\theta}_f(t_2, z)\ \bar{s}^{N-1}_{\kappa,\lambda}(t_1) \leq \bar{\theta}_f(t_1, z)\ \bar{s}^{N-1}_{\kappa,\lambda}(t_2).$$

Integrate the both sides over $[0, r]$ with respect to t_1, and then do that over $[r, R]$ with respect to t_2. It follows that

$$\frac{\int_r^R \bar{\theta}_f(t_2, z)\, dt_2}{\int_0^r \bar{\theta}_f(t_1, z)\, dt_1} \leq \frac{s_{N,\kappa,\lambda}(R) - s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(r)}.$$

By Lemma 8.7, we have

$$\begin{aligned}\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} &= 1 + \frac{\int_{\partial M} \int_r^R \bar{\theta}_f(t_2, z)\, dt_2\, d\,\mathrm{vol}_h}{\int_{\partial M} \int_0^r \bar{\theta}_f(t_1, z)\, dt_1\, d\,\mathrm{vol}_h} \\ &\leq 1 + \frac{s_{N,\kappa,\lambda}(R) - s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(r)} = \frac{s_{N,\kappa,\lambda}(R)}{s_{N,\kappa,\lambda}(r)}.\end{aligned}$$

This completes the proof. ◻

Under the curvature bound (1.4), we have the following:

Theorem 8.16 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. For $N \in (-\infty, 1]$, suppose that we have $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. Then for all $r, R > 0$ with $r \leq R$*

$$\frac{m_{\frac{n+1}{n-1}f}(B_{R,f}(\partial M))}{m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M))} \leq \frac{s_{n,\kappa,\lambda}(R)}{s_{n,\kappa,\lambda}(r)}. \tag{8.16}$$

Proof. Using Lemma 8.2, for all $s_1, s_2 \geq 0$ with $s_1 \leq s_2$ we see

$$\check{\theta}_f(s_2, z)\ \bar{s}^{n-1}_{\kappa,\lambda}(s_1) \leq \check{\theta}_f(s_1, z)\ \bar{s}^{n-1}_{\kappa,\lambda}(s_2).$$

We integrate the both sides over $[0, r]$ with respect to s_1, and then do that over $[r, R]$ with respect to s_2. It follows that

$$\frac{\int_r^R \check{\theta}_f(s_2, z)\, ds_2}{\int_0^r \check{\theta}_f(s_1, z)\, ds_1} \leq \frac{s_{n,\kappa,\lambda}(R) - s_{n,\kappa,\lambda}(r)}{s_{n,\kappa,\lambda}(r)}.$$

Lemma 8.9 tells us that

$$\frac{m_{\frac{n+1}{n-1}f}(B_{R,f}(\partial M))}{m_{\frac{n+1}{n-1}f}(B_{r,f}(\partial M))} = 1 + \frac{\int_{\partial M} \int_r^R \check{\theta}_f(s_2, z)\, ds_2\, d\,\mathrm{vol}_h}{\int_{\partial M} \int_0^r \check{\theta}_f(s_1, z)\, ds_1\, d\,\mathrm{vol}_h} \leq \frac{s_{n,\kappa,\lambda}(R)}{s_{n,\kappa,\lambda}(r)}.$$

We complete the proof. ◻

If f is bounded from above, then we have the following:

Theorem 8.17 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Then for all $r, R > 0$ with $r \leq R$*

$$\frac{m_f(B_{R,\delta}(\partial M))}{m_f(B_{r,\delta}(\partial M))} \leq \frac{s_{n,\kappa,\lambda}(R)}{s_{n,\kappa,\lambda}(r)}. \tag{8.17}$$

Proof. Lemma 8.3 implies that for all $s_1, s_2 \geq 0$ with $s_1 \leq s_2$,

$$\check{\theta}_{f,\delta}(s_2, z)\ \bar{s}^{n-1}_{\kappa,\lambda}(s_1) \leq \check{\theta}_{f,\delta}(s_1, z)\ \bar{s}^{n-1}_{\kappa,\lambda}(s_2).$$

By integrating the both sides over $[0, r]$ with respect to s_1, and then doing that over $[r, R]$ with respect to s_2, we have

$$\frac{\int_r^R \check{\theta}_{f,\delta}(s_2, x)\, ds_2}{\int_0^r \check{\theta}_{f,\delta}(s_1, x)\, ds_1} \leq \frac{s_{n,\kappa,\lambda}(R) - s_{n,\kappa,\lambda}(r)}{s_{n,\kappa,\lambda}(r)}.$$

From Lemma 8.10 we deduce

$$\frac{m_f(B_{R,\delta}(\partial M))}{m_f(B_{r,\delta}(\partial M))} = 1 + \frac{\int_{\partial M} \int_r^R \check{\theta}_{f,\delta}(s_2, z)\, ds_2\, d\,\mathrm{vol}_h}{\int_{\partial M} \int_0^r \check{\theta}_{f,\delta}(s_1, z)\, ds_1\, d\,\mathrm{vol}_h} \leq \frac{s_{n,\kappa,\lambda}(R)}{s_{n,\kappa,\lambda}(r)}.$$

We obtain the desired inequality. □

Furthermore, we show the following:

Theorem 8.18 ([47]). *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let us assume that κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. Then for all $r, R > 0$ with $r \leq R$ we have*

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} \leq \frac{R}{r}. \tag{8.18}$$

Proof. By Lemma 8.5, for all $t_1, t_2 \geq 0$ with $t_1 \leq t_2$

$$\bar{\theta}_f(t_2, z) \leq \bar{\theta}_f(t_1, z).$$

Integrating the both sides over $[0, r]$ with respect to t_1, and then do that over $[r, R]$ with respect to t_2, we see

$$\frac{\int_r^R \bar{\theta}_f(t_2, z)\, dt_2}{\int_0^r \bar{\theta}_f(t_1, z)\, dt_1} \leq \frac{R - r}{r}.$$

Lemma 8.7 implies

$$\frac{m_f(B_R(\partial M))}{m_f(B_r(\partial M))} = 1 + \frac{\int_{\partial M} \int_r^R \bar{\theta}_f(t_2, z)\, dt_2\, d\,\mathrm{vol}_h}{\int_{\partial M} \int_0^r \bar{\theta}_f(t_1, z)\, dt_1\, d\,\mathrm{vol}_h} \leq \frac{R}{r}.$$

We obtain the relative volume comparison theorem. □

8.5 Volume growth rigidity

To prove Theorem 1.6, we show the following:

Lemma 8.19 ([46]). *Under the same setting as in Theorem 8.15, suppose that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in* (8.15) *holds. Then we have $\tau \geq R$ on ∂M.*

Proof. The proof is by contradiction. Suppose that there exists $z_0 \in \partial M$ such that $\tau(z_0) < R$. Put $t_0 := \tau(z_0)$, and take $\epsilon > 0$ with $t_0 + \epsilon < R$. By Lemma 3.4, there exists a closed geodesic ball B in ∂M centered at z_0 such that for all $z \in B$ we have $\tau(z) \leq t_0 + \epsilon$. By Lemma 8.1, we see that $m_f(B_R(\partial M))$ is not larger than

$$\int_{\partial M \setminus B} \int_0^{\min\{R,\tau(z)\}} s_{\kappa,\lambda}^{N-1}(t)\, dt\, d\, m_{f,\partial M} + \int_B \int_0^{t_0+\epsilon} s_{\kappa,\lambda}^{N-1}(t)\, dt\, d\, m_{f,\partial M}.$$

This is smaller than $m_{f,\partial M}(\partial M)\, s_{N,\kappa,\lambda}(R)$. On the other hand, $s_{N,\kappa,\lambda}(R)$ is equal to $m_f(B_R(\partial M))/m_{f,\partial M}(\partial M)$. This is a contradiction. $\square$

We prove Theorem 1.6.

Proof of Theorem 1.6. Suppose that ∂M is compact. For $N \in [n, \infty)$, suppose $\mathrm{Ric}_{f,M}^N \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. We assume (1.18).

By Lemma 8.11 and Theorem 8.15, for all $r, R > 0$ with $r \leq R$, we have

$$\frac{m_f(B_R(\partial M))}{s_{N,\kappa,\lambda}(R)} = \frac{m_f(B_r(\partial M))}{s_{N,\kappa,\lambda}(r)} = m_{f,\partial M}(\partial M). \tag{8.19}$$

If κ and λ satisfy the ball-condition, then for $R = C_{\kappa,\lambda}$, and for every $r \in (0, R]$ the equality in (8.15) holds; in particular, by Lemma 8.19 we have $\tau = C_{\kappa,\lambda}$ on ∂M. If κ and λ do not satisfy the ball-condition, then for every $R > 0$, and for every $r \in (0, R]$ the equality in (8.16) holds; in particular, Lemma 8.19 implies $\tau = \infty$ on ∂M. We obtain $\tau = \bar{C}_{\kappa,\lambda}$ on ∂M.

If κ and λ satisfy the ball-condition, then Lemma 3.2 implies that $D(M, \partial M)$ is equal to $C_{\kappa,\lambda}$. By Lemma 3.6, M is compact; in particular, there exists $x_0 \in M$ such that $\rho_{\partial M}(x_0) = C_{\kappa,\lambda}$. Due to Theorem 1.1, M is isometric to $B^n_{\kappa,\lambda}$, and $N = n$.

If κ and λ do not satisfy the ball-condition, then we see that $\mathrm{Cut}\, \partial M$ is empty. Fix $z \in \partial M$. For all $t \geq 0$ we see $\theta_f(t, z) = e^{-f(z)}\, s_{\kappa,\lambda}^{N-1}(t)$. Let us choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we have $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0, \infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = f(z) - (N-n) \log s_{\kappa,\lambda}$ on $[0, \infty)$ (see Remark 8.1). Define a map $\Phi : [0, \infty) \times \partial M \to M$ by $\Phi(t, z) = \gamma_z(t)$. By the rigidity of Jacobi fields, Φ is a Riemannian isometry with boundary from $[0, \infty) \times_{\kappa,\lambda} \partial M$ to M. This completes the proof of Theorem 1.6. $\square$

For the proof of Theorem 1.7, we show the following:

Lemma 8.20 ([48]). *Under the same setting as in Theorem 8.16, suppose that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in* (8.16) *holds. Then we have $\tau_f \geq R$ on ∂M.*

Proof. We prove by contradiction. Suppose that $z_0 \in \partial M$ satisfies $\tau_f(z_0) < R$. Put $t_0 := \tau_f(z_0)$, and take $\epsilon > 0$ satisfying $t_0 + \epsilon < R$. By using Lemma 3.4, there exists a closed geodesic ball B in ∂M centered at z_0 such that for all $z \in B$ we have $\tau_f(z) \leq t_0 + \epsilon$. From Lemma 8.2 we deduce

$$\begin{aligned} & m_{\frac{n+1}{n-1}f}(B_{R,f}(\partial M)) \\ & \quad \leq \int_{\partial M \setminus B} \int_0^{\min\{R,\tau_f(z)\}} s_{\kappa,\lambda}^{n-1}(s)\, ds\, d\, m_{f,\partial M} + \int_B \int_0^{t_0+\epsilon} s_{\kappa,\lambda}^{n-1}(s)\, ds\, d\, m_{f,\partial M} \\ & \quad < m_{f,\partial M}(\partial M)\, s_{n,\kappa,\lambda}(R). \end{aligned}$$

Since $s_{n,\kappa,\lambda}(R) = m_{\frac{n+1}{n-1}f}(B_{R,f}(\partial M))/m_{f,\partial M}(\partial M)$, this is a contradiction. $\square$

We prove Theorem 1.7.

Proof of Theorem 1.7. Suppose that ∂M is compact. Suppose that κ and λ do not satisfy the ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}_{f,M}^N \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. We assume (1.21).

Lemma 8.12 and Theorem 8.16 imply that for every $R > 0$, and for every $r \in (0, R]$ the equality in (8.16) holds. From Lemma 8.20 we deduce $\tau_f = \infty$ on ∂M. Hence $\tau = \infty$ on ∂M.

We see $\mathrm{Cut}\,\partial M = \emptyset$. For a fixed point $z \in \partial M$, and for all $s \geq 0$ we see $\check{\theta}_f(s, z) = e^{-f(z)}\, s_{\kappa,\lambda}^{n-1}(s)$. We choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we have $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0, \infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0, \infty)$ (see Remark 8.2). Define a map $\Phi : [0, \infty) \times \partial M \to M$ by $\Phi(t, z) = \gamma_z(t)$. We see that Φ is a Riemannian isometry with boundary from $[0, \infty) \times_{F_{\kappa,\lambda}} \partial M$ to M. We complete the proof. $\square$

Remark 8.7. Under the same setting as in Theorem 1.7, if κ and λ satisfy the ball-condition, then Lemmas 4.8 and 8.20 imply $\tau_f = C_{\kappa,\lambda}$ on ∂M. For each $z \in \partial M$ the value $\tau(z)$ can be either finite or infinite, and hence it seems to be difficult to conclude any rigidity results.

Furthermore, we show the following lemma:

Lemma 8.21 ([48])**.** *Under the same setting as in Theorem 8.17, suppose that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for every $r \in (0, R]$ the equality in* (8.17) *holds. Then we have $\tau_\delta \geq R$ on ∂M.*

Proof. The proof is done by contradiction. Let $z_0 \in \partial M$ satisfy $\tau_\delta(z_0) < R$. Put $t_0 := \tau_\delta(z_0)$, and take $\epsilon > 0$ with $t_0 + \epsilon < R$. From Lemma 3.4, it follows that there exists a closed geodesic ball B in ∂M centered at z_0 such that $\tau_\delta \leq t_0 + \epsilon$ on B. By Lemma 8.3, we have

$$\begin{aligned} & e^{-2\delta} m_f(B_{R,\delta}(\partial M)) \\ & \quad \leq \int_{\partial M \setminus B} \int_0^{\min\{R,\tau_\delta(z)\}} s_{\kappa,\lambda}^{n-1}(s)\, ds\, d\, m_{f,\partial M} + \int_B \int_0^{t_0+\epsilon} s_{\kappa,\lambda}^{n-1}(s)\, ds\, d\, m_{f,\partial M} \\ & \quad < m_{f,\partial M}(\partial M)\, s_{n,\kappa,\lambda}(R). \end{aligned}$$

This contradicts $s_{n,\kappa,\lambda}(R) = m_f(B_{R,\delta}(\partial M))/e^{2\delta} m_{f,\partial M}(\partial M)$. $\square$

Using Lemma 8.21, we prove the following volume growth rigidity theorem:

Theorem 8.22 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Suppose*

$$\liminf_{r\to\infty} \frac{m_f(B_{r,\delta}(\partial M))}{s_{n,\kappa,\lambda}(r)} \geq e^{2\delta}\, m_{f,\partial M}(\partial M).$$

Then the following hold:

(1) *if κ and λ satisfy the convex-ball-condition, then M is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on M;*

(2) *if $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then M is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$; moreover, the following hold:*

   (a) *if $\kappa = 0$ and $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0,\infty)$;*

   (b) *if $\kappa < 0$, then $f = (n-1)\delta$ on M.*

Proof. By Lemma 8.13 and Theorem 8.17, for all $r, R > 0$ with $r \leq R$,

$$\frac{m_f(B_{R,\delta}(\partial M))}{s_{n,\kappa,\lambda}(R)} = \frac{m_f(B_{r,\delta}(\partial M))}{s_{n,\kappa,\lambda}(r)} = e^{2\delta} m_{f,\partial M}(\partial M). \tag{8.20}$$

If κ and λ satisfy the convex-ball-condition, then for $R = C_{\kappa,\lambda}$, and for every number $r \in (0, R]$ the equality in (8.17) holds; in particular, by Lemma 8.21 we have $\tau_\delta = C_{\kappa,\lambda}$ on ∂M. If $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then for every $R > 0$, and for every $r \in (0, R]$ the equality in (8.16) holds; in particular, Lemma 8.21 implies $\tau_\delta = \infty$ on ∂M. We obtain $\tau_\delta = \bar{C}_{\kappa,\lambda}$ on ∂M.

If κ and λ satisfy the convex-ball-condition, then Lemma 3.2 tells us that $D_\delta(M, \partial M)$ is equal to $C_{\kappa,\lambda}$. Lemma 3.6 implies that M is compact; in particular, there exists $x_0 \in M$ such that $\rho_{\partial M,\delta}(x_0) = C_{\kappa,\lambda}$. Due to Theorem 1.3, M is isometric to $B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on M.

If $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$, then $\mathrm{Cut}\,\partial M = \emptyset$. Theorem 1.5 tells us that M is isometric to $[0,\infty) \times_{F_{\kappa,\lambda}} \partial M$. Moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0,\infty)$. In the case of $\kappa < 0$, (8.20) implies that for all $s \geq 0$ and $z \in \partial M$ we have $\hat{\theta}_{f,\delta}(s,z) = e^{-f(z)} s^{n-1}_{\kappa,\lambda}(s)$; in particular, $f = (n-1)\delta$ on M (see Remarks 5.4 and 8.3). This completes the proof of Theorem 8.22. $\Box$

We show the following:

Lemma 8.23 ([47]). *Under the same setting as in Theorem 8.18, suppose that there exists $R > 0$ such that for every $r \in (0, R]$ the equality in* (8.18) *holds. Then we have $\tau \geq R$ on ∂M.*

Proof. The proof is done by contradiction. Suppose that for some point $z_0 \in \partial M$ we have $\tau(z_0) < R$. Put $t_0 := \tau(z_0)$, and take $\epsilon > 0$ with $t_0 + \epsilon < R$. By Lemma

3.4, there exists a closed geodesic ball B in ∂M centered at z_0 such that $\tau \leq t_0 + \epsilon$ on B. Lemma 8.5 implies

$$m_f(B_R(\partial M)) \leq R\, m_{f,\partial M}(\partial M \setminus B) + (t_0 + \epsilon)\, m_{f,\partial M}(B) < R\, m_{f,\partial M}(\partial M).$$

On the other hand, $m_f(B_R(\partial M))/m_{f,\partial M}(\partial M) = R$. This is a contradiction. $\square$

We prove the following volume growth rigidity theorem:

Theorem 8.24 ([47])**.** *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let us assume that κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. If we have*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{r} \geq m_{f,\partial M}(\partial M),$$

then M is isometric to $[0, \infty) \times_{F_{0,0}} \partial M$; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0, \infty)$; in particular, M is isometric to $[0, \infty) \times \partial M$.

Proof. By Lemma 8.14 and Theorem 8.18, for all $r, R > 0$ with $r \leq R$, we have

$$\frac{m_f(B_R(\partial M))}{R} = \frac{m_f(B_r(\partial M))}{r} = m_{f,\partial M}(\partial M). \tag{8.21}$$

For every $R > 0$, and for every $r \in (0, R]$ the equality in (8.18) holds; in particular, Lemma 8.23 implies $\tau = \infty$ on ∂M.

Since $\tau = \infty$ on ∂M, we have $\mathrm{Cut}\,\partial M = \emptyset$. Fix $z \in \partial M$. For all $t \geq 0$ we see $\theta_f(t, z) = e^{-f(z)}$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we have $Y_{z,i} = F_{0,0}\, E_{z,i}$ on $[0, \infty)$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty, 1)$, then $f \circ \gamma_z$ is constant on $[0, \infty)$ (see Remark 8.5). Define a map $\Phi : [0, \infty) \times \partial M \to M$ by $\Phi(t, z) = \gamma_z(t)$. The rigidity of Jacobi fields imply that Φ is a Riemannian isometry with boundary from $[0, \infty) \times_{F_{0,0}} \partial M$ to M. This proves Theorem 8.24. $\square$

By Proposition 2.10, we have the following corollary of Theorem 8.24.

Corollary 8.25 ([47])**.** *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Suppose $\mathrm{Ric}^1_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. If we have*

$$\liminf_{r\to\infty} \frac{m_f(B_r(\partial M))}{r} \geq m_{f,\partial M}(\partial M),$$

then there exist a function $f_0 : [0, \infty) \to \mathbb{R}$, and a Riemannian metric h_0 on ∂M such that M is isometric to a warped product $\left([0, \infty) \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0\right)$.

Chapter 9

Eigenvalue rigidity

In this chapter, we give lower bounds for the smallest Dirichlet eigenvalues for the weighted p-Laplacian, and study rigidity theorems in the equality cases.

9.1 Lower bounds

We show the following inequality of Picone type that has been obtained by Allegretto and Huang [1] in a Euclidean setting (see Theorem 1.1 in [1]):

Lemma 9.1 ([1], [46])**.** *Let ϕ and ψ be functions on M that are smooth on a domain U in M, and satisfy $\phi > 0$ and $\psi \geq 0$ on U. Then for all $p \in (1,\infty)$ we have the following inequality on U:*

$$\|\nabla\psi\|^p \geq \|\nabla\phi\|^{p-2} g\left(\nabla\left(\psi^p\,\phi^{1-p}\right), \nabla\phi\right). \tag{9.1}$$

Proof. For $p \in (1,\infty)$, put $q := p(p-1)^{-1}$. From the Young inequality we deduce

$$\|\nabla\psi\| \left(\frac{\psi\|\nabla\phi\|}{\phi}\right)^{p-1} \leq \frac{\|\nabla\psi\|^p}{p} + \frac{1}{q}\left(\frac{\psi\|\nabla\phi\|}{\phi}\right)^{p} \tag{9.2}$$

on U. Furthermore, by (9.2) and the Cauchy-Schwarz inequality, we see

$$\begin{aligned}\|\nabla\psi\|^p &\geq p\left(\psi\phi^{-1}\right)^{p-1}\|\nabla\psi\|\|\nabla\phi\|^{p-1} - (p-1)\left(\psi\phi^{-1}\right)^p\|\nabla\phi\|^p \\ &\geq p\left(\psi\phi^{-1}\right)^{p-1} g(\nabla\phi,\nabla\psi)\|\nabla\phi\|^{p-2} - (p-1)\left(\psi\phi^{-1}\right)^p\|\nabla\phi\|^p \\ &= \|\nabla\phi\|^{p-2} g\left(\nabla\left(\psi^p\,\phi^{1-p}\right),\nabla\phi\right),\end{aligned} \tag{9.3}$$

and hence (9.1). ☐

Remark 9.1. Assume that the equality in (9.1) holds on U. Then the equalities in (9.3) also hold. By the equality in the Young inequality, and by that in the Cauchy-Schwarz inequality, we see that for some constant $c \neq 0$ we have $\phi\|\nabla\psi\| = \psi\|\nabla\phi\|$ and $\nabla\psi = c\nabla\phi$ on U; in particular, $\psi = c\,\phi$ on U.

From Lemma 9.1 we derive the inequality (1.24) in Theorem 1.8.

Proposition 9.2 ([46])**.** *Let M be compact. Let $p \in (1,\infty)$. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0,\bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D(M,\partial M) \leq D$. Then we have* (1.24).

Proof. Let $\hat{\varphi} : [0, D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,N,\kappa,\lambda,D}$. We may assume $\hat{\varphi}|_{(0,D]} > 0$. The equation (1.23) is written in the form

$$\left(|\varphi'(t)|^{p-2}\varphi'(t)s_{\kappa,\lambda}^{N-1}(t)\right)' + \mu\,|\varphi(t)|^{p-2}\varphi(t)s_{\kappa,\lambda}^{N-1}(t) = 0,\ \varphi(0) = 0,\ \varphi'(D) = 0.$$

Hence we see $\hat{\varphi}'|_{[0,D)} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\operatorname{Int} M$. From Lemma 9.1 we deduce

$$\|\nabla\psi\|^p \geq \|\nabla\Phi\|^{p-2} g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right), \nabla\Phi\right) \tag{9.4}$$

on $\operatorname{Int} M \setminus \operatorname{Cut}\partial M$. Using (9.4) and Proposition 5.2, we obtain

$$\begin{aligned}
&\int_M \|\nabla\psi\|^p\, d\,m_{\hat{f}}\\
&\quad \geq -\int_M \psi^p\,\Phi^{1-p}\left\{\left(\left(\left(\hat{\varphi}'\right)^{p-1}\right)' - H_{N,\kappa,\lambda}\,\left(\hat{\varphi}'\right)^{p-1}\right)\circ\rho_{\partial M}\right\} dm_{\hat{f}}\\
&\quad = \mu_{p,N,\kappa,\lambda,D}\int_M \psi^p\, d\,m_{\hat{f}}.
\end{aligned}$$

Therefore, $R_{f,p}(\psi) \geq \mu_{p,N,\kappa,\lambda,D}$. This implies (1.24). □

Remark 9.2. In Proposition 9.2, we assume that there exists a non-negative, non-zero smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\operatorname{Int} M$ such that $R_{f,p}(\psi) = \mu_{p,N,\kappa,\lambda,D}$. Then the equality in (9.4) holds on $\operatorname{Int} M \setminus \operatorname{Cut}\partial M$. Hence for some constant $c \neq 0$ we have $\psi = c\,\Phi$ on M (see Remark 9.1); moreover, we have the equality in Proposition 5.2 (see Remark 5.2).

Next, we prove the inequality (1.26) in Theorem 1.9.

Proposition 9.3 ([48])**.** *Let M be compact, and let f be ∂M-radial. Let $p \in (1,\infty)$. For $N \in (-\infty, 1]$, we suppose $\operatorname{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, we suppose $f \leq (n-1)\delta$ on M. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D_f(M, \partial M) \leq D$, where $D_f(M, \partial M)$ is defined as* (1.25)*. Then we have* (1.26)*.*

Proof. Let $\hat{\varphi} : [0, D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,n,\kappa,\lambda,D}$, and let $\hat{\varphi}|_{(0,D]} > 0$. We see $\hat{\varphi}'|_{[0,D)} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M,f}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\operatorname{Int} M$. By Lemma 9.1, we have

$$\|\nabla\psi\|^p \geq \|\nabla\Phi\|^{p-2} g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right), \nabla\Phi\right) \tag{9.5}$$

on $\operatorname{Int} M \setminus \operatorname{Cut}\partial M$. We now put

$$\hat{f} := \frac{n+1}{n-1} f, \quad \check{f} := \frac{n+1-2p}{n-1} f.$$

By $f \leq (n-1)\delta$ and (9.5),

$$\begin{aligned}
e^{2p\delta}\int_M \|\nabla\psi\|^p\, d\,m_{\hat{f}} &\geq \int_M e^{\frac{2pf}{n-1}}\|\nabla\psi\|^p\, d\,m_{\hat{f}} = \int_M \|\nabla\psi\|^p\, d\,m_{\check{f}} \\
&\geq \int_M \|\nabla\Phi\|^{p-2} g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right), \nabla\Phi\right)\, d\,m_{\check{f}}.
\end{aligned} \tag{9.6}$$

The inequality (9.6) and Proposition 5.7 imply

$$\begin{aligned}
&e^{2p\delta}\int_M \|\nabla\psi\|^p\, d\, m_{\hat{f}}\\
&\qquad \geq -\int_M \psi^p\,\Phi^{1-p}\left\{\left(\left((\hat{\varphi}')^{p-1}\right)'-H_{n,\kappa,\lambda}\,(\hat{\varphi}')^{p-1}\right)\circ\rho_{\partial M,f}\right\}dm_{\hat{f}}\\
&\qquad = \mu_{p,n,\kappa,\lambda,D}\int_M \psi^p\, d\, m_{\hat{f}}.
\end{aligned}$$

It follows that $R_{\hat{f},p}(\psi)\geq e^{-2p\delta}\mu_{p,n,\kappa,\lambda,D}$. This leads us to (1.26). $\square$

Remark 9.3. In Proposition 9.3, we assume that there exists a non-negative, non-zero smooth function $\psi : M\to\mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$ such that $R_{\frac{n+1}{n-1}f,p}(\psi)=e^{-2p\delta}\mu_{p,n,\kappa,\lambda,D}$. Then the equalities in (9.6) hold. Since the equality in (9.5) holds on $\mathrm{Int}\, M\setminus\mathrm{Cut}\,\partial M$, for some constant $c\neq 0$ we have $\psi=c\,\Phi$ on M (see Remark 9.1). Moreover, we see $f=(n-1)\delta$ on the set where $\nabla\psi\neq 0$.

We also prove the inequality (1.27) in Theorem 1.10:

Proposition 9.4 ([48])**.** *Suppose that M is compact. Let $p\in(1,\infty)$. Let κ and λ satisfy the convex-ball-condition. For $N\in(-\infty,1]$, suppose $\mathrm{Ric}^N_{f,M}\geq(n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M}\geq(n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta\in\mathbb{R}$, suppose $f\leq(n-1)\delta$. Then we have* (1.27).

Proof. A standard argument implies $\mu_{0,p}(B^n_{\kappa,\lambda})=\mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$. Let $\hat{\varphi}:[0,C_{\kappa,\lambda}]\to\mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu=\mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$, and let $\hat{\varphi}|_{(0,C_{\kappa,\lambda}]}>0$. It holds that $\hat{\varphi}'|_{[0,C_{\kappa,\lambda})}>0$. By Proposition 6.3, we have $D_\delta(M,\partial M)\leq C_{\kappa,\lambda}$. Put $\Phi:=\hat{\varphi}\circ\rho_{\partial M,\delta}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\mathrm{Int}\, M$. By Lemma 9.1,

$$\|\nabla\psi\|^p\geq\|\nabla\Phi\|^{p-2}g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right),\nabla\Phi\right)\tag{9.7}$$

on $\mathrm{Int}\, M\setminus\mathrm{Cut}\,\partial M$. We notice that κ and λ satisfy the model-condition. From (9.7) and Proposition 5.5, it follows that

$$\begin{aligned}
&e^{2p\delta}\int_M\|\nabla\psi\|^p\, d\, m_f\geq e^{2p\delta}\int_M\|\nabla\Phi\|^{p-2}g\left(\nabla\left(\psi^p\,\Phi^{1-p}\right),\nabla\Phi\right)\,d\,m_f\\
&\qquad\geq-\int_M\psi^p\,\Phi^{1-p}\left\{\left(\left((\hat{\varphi}')^{p-1}\right)'-H_{n,\kappa,\lambda}\,(\hat{\varphi}')^{p-1}\right)\circ\rho_{\partial M,\delta}\right\}d\,m_f\\
&\qquad=\mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}\int_M\psi^p\, d\, m_f.
\end{aligned}$$

We obtain $R_{f,p}(\psi)\geq e^{-2p\delta}\mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$. This proves the proposition. $\square$

Remark 9.4. In Proposition 9.4, we assume that there exists a non-negative, non-zero smooth function $\psi : M\to\mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$ such that $R_{f,p}(\psi)=e^{-2p\delta}\mu_{0,p}(B^n_{\kappa,\lambda})$. Then the equality in (9.7) holds on $\mathrm{Int}\, M\setminus\mathrm{Cut}\,\partial M$. Hence for some constant $c\neq 0$ we have $\psi=c\,\Phi$ on M (see Remark 9.1); moreover, we have the equality in Proposition 5.5 (see Remark 5.6).

Furthermore, by Proposition 5.9, we have the following:

Proposition 9.5 ([47]). *Suppose that M is compact. Let $p \in (1,\infty)$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. For $D > 0$, suppose $D(M, \partial M) \leq D$. Then $\mu_{f,p}(M) \geq \mu_{p,n,0,0,D}$.*

Proof. Let $\hat{\varphi} : [0, D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,n,0,0,D}$, and let $\hat{\varphi}|_{(0,D]} > 0$. We see $\hat{\varphi}'|_{[0,D)} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M}$. Take a non-negative, non-zero smooth function ψ on M whose support is compact and contained in $\mathrm{Int}\, M$. By Lemma 9.1

$$\|\nabla\psi\|^p \geq \|\nabla\Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla\Phi\right) \tag{9.8}$$

on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. From (9.8) and Proposition 5.9 we derive

$$\begin{aligned}\int_M \|\nabla\psi\|^p \, dm_f &\geq \int_M \|\nabla\Phi\|^{p-2} g\left(\nabla\left(\psi^p\, \Phi^{1-p}\right), \nabla\Phi\right) \, dm_f \\ &\geq -\int_M \psi^p\, \Phi^{1-p} \left(\left(\left(\hat{\varphi}'\right)^{p-1}\right)' \circ \rho_{\partial M,\delta}\right) dm_f \\ &= \mu_{p,n,0,0,D} \int_M \psi^p \, dm_f.\end{aligned}$$

This implies $R_{f,p}(\psi) \geq \mu_{p,n,0,0,D}$. We arrive at the desired inequality. $\square$

Remark 9.5. In Proposition 9.5, we assume that there exists a non-negative, non-zero smooth function $\psi : M \to \mathbb{R}$ whose support is compact and contained in $\mathrm{Int}\, M$ such that $R_{f,p}(\psi) = \mu_{p,n,0,0,D}$. Then the equality in (9.8) holds on $\mathrm{Int}\, M \setminus \mathrm{Cut}\, \partial M$. Hence for some constant $c \neq 0$ we have $\psi = c\,\Phi$ on M (see Remark 9.1); moreover, we have the equality in Proposition 5.9 (see Remark 5.8).

9.2 Equality cases

We recall the following fact for eigenfunctions of the weighted p-Laplacian.

Proposition 9.6 ([46], [51]). *Let $\phi : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1,\infty)$. Then there exists a non-negative, non-zero function ψ in $W^{1,p}_0(M, m_\phi)$ such that $R_{\phi,p}(\psi) = \mu_{\phi,p}(M)$. Moreover, for some $\alpha \in (0,1)$ the function ψ is $C^{1,\alpha}$-Hölder continuous.*

Proposition 9.6 is well-known in the case of $\phi = 0$. In that case, the existence follows from the standard compactness argument, and the regularity follows from the results by Tolksdorf in [51]. The method of the proof also works in our setting.

For $D > 0$, we put $S_D(\partial M) := \{\, x \in M \mid \rho_{\partial M}(x) = D \,\}$.

Kasue [25] has shown the following in the proof of Theorem 2.1 in [25]:

Proposition 9.7 ([25]). *Let M be compact. Suppose that for some $D \in (0, \bar{C}_{\kappa,\lambda})$ we have $\mathrm{Cut}\, \partial M = S_D(\partial M)$. For each $z \in \partial M$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. Assume further that for all $z \in \partial M$ and i we have $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0, D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Then κ and λ satisfy the model-condition, M is isometric to a (κ,λ)-equational model space, and $D = D_{\kappa,\lambda}(M)$.*

We now prove Theorem 1.8

Proof of Theorem 1.8. Let M be compact. Let $p \in (1,\infty)$. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0,\bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D(M,\partial M) \leq D$. By Proposition 9.2, we have (1.24).

Assume that the equality in (1.24) holds. By applying Proposition 9.6 to the function f, there exists a non-negative, non-zero function ψ in $W^{1,p}_0(M,m_f)$ such that $R_{f,p}(\psi) = \mu_{p,N,\kappa,\lambda,D}$, and ψ is $C^{1,\alpha}$-Hölder continuous on M. Let $\hat{\varphi} : [0,D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,N,\kappa,\lambda,D}$, and let $\hat{\varphi}|_{(0,D]} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M}$. Then Φ coincides with a constant multiplication of ψ on M (see Remark 9.2); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous.

Fix a point $z \in \partial M$. Choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we see $Y_{z,i} = s_{\kappa,\lambda}\, E_{z,i}$ on $[0,\tau(z)]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = f(z) - (N-n)\log s_{\kappa,\lambda}$ on $[0,\tau(z)]$ (see Remark 9.2).

Let $D = \bar{C}_{\kappa,\lambda}$. Since D is finite, κ and λ satisfy the ball-condition and $D = C_{\kappa,\lambda}$. There exists $x_0 \in M$ such that $\rho_{\partial M}(x_0) = D(M,\partial M)$. Note that x_0 belongs to $\mathrm{Cut}\,\partial M$. Now, we prove $\rho_{\partial M}(x_0) = C_{\kappa,\lambda}$. We assume $\rho_{\partial M}(x_0) < C_{\kappa,\lambda}$. Let z_0 be a foot point on ∂M of x_0. From the property of Jacobi fields, x_0 is not the first conjugate point of ∂M along γ_{z_0}. Hence $\rho_{\partial M}$ is not differentiable at x_0. Since Φ is $C^{1,\alpha}$-Hölder continuous, we see $\hat{\varphi}(\rho_{\partial M}(x_0)) = 0$. From $\hat{\varphi}|_{[0,D)} > 0$ we deduce $\rho_{\partial M}(x_0) = D$. This contradicts $D = C_{\kappa,\lambda}$. Therefore, $\rho_{\partial M}(x_0) = C_{\kappa,\lambda}$. By Theorem 1.1, M is isometric to $B^n_{\kappa,\lambda}$ and $N = n$.

Let $D \in (0,\bar{C}_{\kappa,\lambda})$. We prove $\mathrm{Cut}\,\partial M = S_D(\partial M)$. Since $D(M,\partial M) \leq D$, we see $S_D(\partial M) \subset \mathrm{Cut}\,\partial M$. We show the opposite. Take $x_0 \in \mathrm{Cut}\,\partial M$. By the property of Jacobi fields, $\rho_{\partial M}$ is not differentiable at x_0. The regularity of Φ implies $\hat{\varphi}(\rho_{\partial M}(x_0)) = 0$; in particular, $\rho_{\partial M}(x_0) = D$. We have $\mathrm{Cut}\,\partial M = S_D(\partial M)$. By Proposition 9.7, κ and λ satisfy the model-condition, M is isometric to a (κ,λ)-equational model space, and $D = D_{\kappa,\lambda}(M)$. From $\tau = D_{\kappa,\lambda}(M)$ on ∂M, it follows that $f \circ \gamma_z = f(z) - (N-n)\log s_{\kappa,\lambda}$ on $[0, D_{\kappa,\lambda}(M)]$ for all $z \in \partial M$. We complete the proof of Theorem 1.8. $\square$

Remark 9.6. In the case where $f = 0, N = n$ and $p = 2$, Kasue [25] has proved Theorem 1.8 relying on the approximation theorem obtained by Greene and Wu [16]. It seems that the approximation theorem in [16] does not work in our non-linear case of $p \neq 2$.

We next prove Theorem 1.9.

Proof of Theorem 1.9. Let M be compact, and let f be ∂M-radial. Let $p \in (1,\infty)$. For $N \in (-\infty,1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, we suppose $f \leq (n-1)\delta$ on M. For $D \in (0,\bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, we suppose $D_f(M,\partial M) \leq D$. By Proposition 9.3, we have (1.26).

Assume that the equality in (1.26) holds. Applying Proposition 9.6 to the function $(n+1)(n-1)^{-1} f$, we see that there exists a non-negative, non-zero function ψ in $W^{1,p}_0(M, m_{\frac{n+1}{n-1}f})$ such that $R_{\frac{n+1}{n-1}f,p}(\psi) = e^{-2p\delta}\mu_{p,n,\kappa,\lambda,D}$ and ψ is $C^{1,\alpha}$-Hölder continuous on M. Let $\hat{\varphi} : [0,D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,n,\kappa,\lambda,D}$, and let $\hat{\varphi}|_{(0,D]} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M,f}$. Then Φ coincides with a constant multiplication of ψ on M (see Remark 9.3). Since $\hat{\varphi}'|_{[0,D)} > 0$, we see $\nabla\Phi \neq 0$ on $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$; in particular, $\nabla\psi \neq 0$ on $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$. We obtain

$f = (n-1)\delta$ on M (see Remark 9.3). For the infimum Ric_M of Ric_g on the unit tangent bundle of M, and for $H_{\partial M} := \inf_{z \in \partial M} H_z$, we have

$$\mathrm{Ric}_M \geq (n-1)\kappa e^{-4\delta}, \quad H_{\partial M} \geq (n-1)\lambda e^{-2\delta}, \quad D(M, \partial M) \leq D e^{2\delta},$$

and

$$\mu_{0,p}(M) = e^{-2p\delta} \mu_{p,n,\kappa,\lambda,D} = \mu_{p,n,\kappa e^{-4\delta}, \lambda e^{-2\delta}, D e^{2\delta}}.$$

By Theorem 1.8, M is isometric to a $(\kappa e^{-4\delta}, \lambda e^{-2\delta})$-equational model space. We complete the proof of Theorem 1.9. $\Box$

Furthermore, we prove Theorem 1.10.

Proof of Theorem 1.10. Suppose that M is compact. Let $p \in (1, \infty)$. Let κ and λ satisfy the convex-ball-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Proposition 9.4 implies (1.27).

Assume that the equality in (1.27) holds. Note that $\mu_{0,p}(B^n_{\kappa,\lambda}) = \mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$. By Proposition 9.6, there exists a non-negative, non-zero function ψ in $W^{1,p}_0(M, m_f)$ such that $R_{f,p}(\psi) = e^{-2p\delta} \mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$, and ψ is $C^{1,\alpha}$-Hölder continuous on M. Let $\hat{\varphi} : [0, C_{\kappa,\lambda}] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,n,\kappa,\lambda,C_{\kappa,\lambda}}$, and let $\hat{\varphi}|_{(0,C_{\kappa,\lambda}]} > 0$. Proposition 6.3 tells us that $D_\delta(M, \partial M) \leq C_{\kappa,\lambda}$. Let $\Phi := \hat{\varphi} \circ \rho_{\partial M,\delta}$. In this case, Φ coincides with a constant multiplication of ψ on M (see Remark 9.4); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous.

Fix $z \in \partial M$. Let us choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we see $Y_{z,i} = F_{\kappa,\lambda,z}\, E_{z,i}$ on $[0, \tau(z)]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, $f \circ \gamma_z = (n-1)\delta$ on $[0, \tau(z)]$ (see Remark 9.4).

We prove $D_\delta(M, \partial M) = C_{\kappa,\lambda}$ by contradiction. Assume $D_\delta(M, \partial M) < C_{\kappa,\lambda}$. Since M is compact, for some $x_0 \in M$ we have $\rho_{\partial M,\delta}(x_0) = D_\delta(M, \partial M)$. The point x_0 belongs to $\mathrm{Cut}\,\partial M$. Take a foot point z_0 on ∂M of x_0. By $\rho_{\partial M,\delta}(x_0) < C_{\kappa,\lambda}$, and by the property of Jacobi fields, x_0 is not the first conjugate point of ∂M along γ_{z_0}. Hence $\rho_{\partial M,\delta}$ is not differentiable at x_0. Since Φ is $C^{1,\alpha}$-Hölder continuous on M, we have $\hat{\varphi}'(\rho_{\partial M,\delta}(x_0)) = 0$. This contradicts $\hat{\varphi}'|_{[0,C_{\kappa,\lambda})} > 0$. We obtain $D_\delta(M, \partial M) = C_{\kappa,\lambda}$. Due to Theorem 1.3, M is isometric to $B^n_{\kappa e^{-4\delta}, \lambda e^{-2\delta}}$, and $f = (n-1)\delta$ on M. We conclude Theorem 1.10. $\Box$

Let us suppose that M is compact. We say that M is isometric to an $F_{0,0}$*-model space* if M is isometric to either (1) for a connected component ∂M_1 of ∂M, the twisted product $[0, 2D(M, \partial M)] \times_{F_{0,0}} \partial M_1$; or (2) for an involutive isometry σ of ∂M without fixed points, the quotient space $\big([0, 2D(M, \partial M)] \times_{F_{0,0}} \partial M\big)/G_\sigma$, where G_σ denotes the isometry group on $[0, 2D(M, \partial M)] \times_{F_{0,0}} \partial M$ of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, z) := (2D(M, \partial M) - t, \sigma(z))$. Now, we notice that if M is isometric to an $F_{0,0}$-model space, and if for every point $z \in \partial M$ the function $f \circ \gamma_z$ is constant on the interval $[0, D(M, \partial M)]$, then M is isometric to a $(0,0)$-equational model space.

For the proof of Theorem 9.9, we show the following:

Lemma 9.8 ([47]). *Let M be compact. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$, and $H_{f,\partial M} \geq \lambda$. For $D > 0$, suppose $\mathrm{Cut}\,\partial M = S_D(\partial M)$. For each $z \in \partial M$, choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z \partial M$, and let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. Assume further that for all i we have $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0, D]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Then M is isometric to an $F_{0,0}$-model space.*

Proof. We first assume that ∂M is disconnected. Let $\{\partial M_i\}_{i=1,2,\dots}$ denote the connected components of ∂M. Put $D_1 := \inf_{i=2,3,\dots} d_M(\partial M_1, \partial M_i)$. By Theorem 7.16, there exists a connected component ∂M_1 such that M is isometric to $[0, D_1] \times_{F_{0,0}} \partial M_1$. From $\mathrm{Cut}\,\partial M = S_D(\partial M)$, it follows that $D(M, \partial M) = D$ and $D_1 = 2D$. We conclude that M is isometric to an $F_{0,0}$-model space.

We next assume that ∂M is connected. By $\mathrm{Cut}\,\partial M = S_D(\partial M)$, we have $D(M, \partial M) = D$. By the property of Jacobi fields, $S_D(\partial M)$ is a smooth hypersurface in M, and every point in $S_D(\partial M)$ has two distinct foot points on ∂M. For every $z \in \partial M$, the vector $\gamma'_z(D)$ is orthogonal to $S_D(\partial M)$. Hence the number of foot points on ∂M of $\gamma_z(D)$ is equal to two. Define an involutive isometry $\sigma : \partial M \to \partial M$ without fixed points by $\sigma(z) := \hat{z}$, where $\hat{z}$ denotes the foot point on ∂M of $\gamma_z(D)$ that is different from z. Furthermore, we define a map $\Phi : [0, 2D] \times \partial M \to M$ as follows: If $t \in [0, D]$, then $\Phi(t, z) := \gamma_z(t)$; if $t \in (D, 2D]$, then $\Phi(t, z) := \gamma_{\sigma(z)}(2D - t)$. We see that Φ is surjective and continuous. For all $z \in \partial M$ and $t \in [0, 2D]$ we have $\Phi(t, z) = \Phi(2D - t, \sigma(z))$. Since for all $z \in \partial M$ and i we have $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0, D]$, the map $\Phi|_{[0,D)}$ gives an isometry between $(U_D(\partial M), g)$ and the twisted product space $[0, D) \times_{F_{0,0}} \partial M$. Therefore, M is isometric to the quotient space $\big([0, 2D] \times_{F_{0,0}} \partial M\big)/G_\sigma$, where G_σ is the isometry group on $[0, 2D] \times_{F_{0,0}} \partial M$ of the identity and the involute isometry $\hat{\sigma}$ defined by $\hat{\sigma}(t, z) := (2D - t, \sigma(z))$. This implies that M is isometric to an $F_{0,0}$-model space. $\square$

For the equality case of Proposition 9.5, we have the following:

Theorem 9.9 ([47]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1, \infty)$. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. For $D > 0$, suppose $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,p}(M) \geq \mu_{p,n,0,0,D}. \tag{9.9}$$

If the equality in (9.9) holds, then we have $D(M, \partial M) = D$, and M is isometric to an $F_{0,0}$-model space; moreover, if $N \in (-\infty, 1)$, then for every point $z \in \partial M$ the function $f \circ \gamma_z$ is constant on the interval $[0, D]$; in particular, M is isometric to a $(0, 0)$-equational model space.

Proof. From Proposition 9.5 we deduce (9.9). Assume that the equality in (9.9) holds. Proposition 9.6 implies that there exists a non-negative, non-zero function ψ in $W^{1,p}_0(M, m_f)$ such that $R_{f,p}(\psi) = \mu_{p,n,0,0,D}$, and ψ is $C^{1,\alpha}$-Hölder continuous on M. Let $\hat{\varphi} : [0, D] \to \mathbb{R}$ be a non-zero function satisfying (1.23) for $\mu = \mu_{p,n,0,0,D}$, and let $\hat{\varphi}|_{(0,D]} > 0$. Put $\Phi := \hat{\varphi} \circ \rho_{\partial M}$. Then Φ coincides with a constant multiplication of ψ on M (see Remark 9.5); in particular, Φ is also $C^{1,\alpha}$-Hölder continuous.

We fix $z \in \partial M$, and choose an orthonormal basis $\{e_{z,i}\}_{i=1}^{n-1}$ of $T_z\partial M$. Let $\{Y_{z,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_z with initial conditions $Y_{z,i}(0) = e_{z,i}$ and $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$. For all i we see $Y_{z,i} = F_{0,0,z}\, E_{z,i}$ on $[0,\tau(z)]$, where $\{E_{z,i}\}_{i=1}^{n-1}$ are the parallel vector fields along γ_z with initial condition $E_{z,i}(0) = e_{z,i}$. Moreover, if $N \in (-\infty,1)$, then $f \circ \gamma_z$ is constant on $[0,\tau(z)]$ (see Remark 9.5).

We show $\mathrm{Cut}\,\partial M = S_D(\partial M)$. From $D(M,\partial M) \leq D$ we deduce that $S_D(\partial M)$ is contained in $\mathrm{Cut}\,\partial M$. We prove the opposite. Take $x_0 \in \mathrm{Cut}\,\partial M$. By the property of Jacobi fields, $\rho_{\partial M}$ is not differentiable at x_0. From the regularity of Φ, it follows that $\hat{\varphi}'(\rho_{\partial M}(x_0)) = 0$; in particular, $\rho_{\partial M}(x_0) = D$. Hence we have $\mathrm{Cut}\,\partial M = S_D(\partial M)$. This implies $D(M,\partial M) = D$. By using Lemma 9.8, we complete the proof of Theorem 9.9. $\square$

Let us suppose that M is compact. We say that M is isometric to a *warped model space* if there exist a function $f_0 : [0, 2D(M,\partial M)] \to \mathbb{R}$, and a Riemannian metric h_0 on ∂M such that M is isometric to either (1) for a connected component ∂M_1 of ∂M, the warped product $\left([0,2D(M,\partial M)] \times \partial M_1, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0\right)$; or (2) for an involutive isometry σ of ∂M without fixed points, the quotient space $\left([0,2D(M,\partial M)] \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0\right)/G_\sigma$, where G_σ is the isometry group on $\left([0,2D(M,\partial M)] \times \partial M, dt^2 + e^{2\frac{f_0(t)}{n-1}} h_0\right)$ of the identity and the involute isometry $\hat{\sigma}$ defined as $\hat{\sigma}(t,z) := (2D(M,\partial M) - t, \sigma(z))$.

By Proposition 2.10, we have the following corollary of Theorem 9.9:

Corollary 9.10 ([47]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let $p \in (1,\infty)$. Suppose $\mathrm{Ric}^1_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $D > 0$, suppose $D(M,\partial M) \leq D$. Then we have*

$$\mu_{f,p}(M) \geq \mu_{p,n,0,0,D}. \tag{9.10}$$

If the equality in (9.10) holds, then M is isometric to a warped model space.

9.3 Explicit lower bounds

By $\mu_{2,N,0,0,D} = \pi^2 (2D)^{-2}$, we have the following corollary of Theorem 1.9:

Corollary 9.11 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact, and f is ∂M-radial. For $N \in (-\infty,1]$, suppose $\mathrm{Ric}^N_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. For $D > 0$, suppose $D_f(M,\partial M) \leq D$. Then*

$$\mu_{\frac{n+1}{n-1} f,2}(M) \geq \frac{\pi^2}{4\, e^{4\delta}\, D^2}. \tag{9.11}$$

If the equality in (9.11) holds, then M is isometric to a $(0,0)$-equational model space, and $f = (n-1)\delta$ on M.

When $f = 0$ and $\delta = 0$, Li and Yau [32] have obtained the estimate (9.11).

Furthermore, we have the following corollary of Theorem 9.9:

Corollary 9.12 ([47]). *Let M be a connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. Let κ and λ satisfy the subharmonic-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq \kappa$ and $H_{f,\partial M} \geq \lambda$. For $D > 0$, suppose $D(M, \partial M) \leq D$. Then*

$$\mu_{f,2}(M) \geq \frac{\pi^2}{4D^2}. \tag{9.12}$$

If the equality in (9.12) holds, then $D(M, \partial M) = D$, and M is isometric to an $F_{0,0}$-model space; moreover, if $N \in (-\infty, 1)$, then for every $z \in \partial M$ the function $f \circ \gamma_z$ is constant on $[0, D]$; in particular, M is isometric to a $(0, 0)$-equational model space.

Kasue [25] has proved the following estimate (see Lemma 1.3 in [25]):

Lemma 9.13 ([25]). *For all $N \in (1, \infty)$ and $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, we have*

$$\mu_{2,N,\kappa,\lambda,D} > \left(4 \max_{a \in [0,D]} \int_a^D s^{N-1}_{\kappa,\lambda}(b)\, db \int_0^a s^{1-N}_{\kappa,\lambda}(b)\, db\right)^{-1}.$$

From Lemma 9.13 we derive the following corollary of Theorem 1.8:

Corollary 9.14 ([46]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that M is compact. For $N \in [n, \infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D(M, \partial M) \leq D$. Then we have*

$$\mu_{f,2}(M) > \left(4 \max_{a \in [0,D]} \int_a^D s^{N-1}_{\kappa,\lambda}(b)\, db \int_0^a s^{1-N}_{\kappa,\lambda}(b)\, db\right)^{-1}.$$

Furthermore, we have the following corollary of Theorem 1.9:

Corollary 9.15 ([48]). *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Let M be compact, and let f be ∂M-radial. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, suppose $D_f(M, \partial M) \leq D$. Then we have*

$$\mu_{\frac{n+1}{n-1}f,2}(M) > \left(4\, e^{4\delta} \max_{a \in [0,D]} \int_a^D s^{n-1}_{\kappa,\lambda}(b)\, db \int_0^a s^{1-n}_{\kappa,\lambda}(b)\, db\right)^{-1}.$$

Chapter 10

Spectrum rigidity

In this chapter, we present lower bounds for the spectrum of the weighted p-Laplacian, and conclude spectrum rigidity theorems. For a relatively compact domain Ω in M whose boundary $\partial\Omega$ is a smooth hypersurface in M, we show upper estimates for the ratio of $m_f(\Omega)$ to $m_{f,\partial\Omega}(\partial\Omega)$, where for the canonical Riemannian volume measure $\mathrm{vol}_{\partial\Omega}$ on $\partial\Omega$, we put $m_{f,\partial\Omega} := e^{-f|_{\partial\Omega}}\,\mathrm{vol}_{\partial\Omega}$. From the area estimates and the relationship between the isoperimetric constant and the Sobolev constant we derive the lower bounds for the spectrum.

10.1 Area estimates

We prove the following area estimate:

Proposition 10.1 ([46]). *For $N \in [n,\infty)$, suppose that we have $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. We put*

$$D_1(\Omega) := \inf_{x\in\Omega} \rho_{\partial M}(x), \quad D_2(\Omega) := \sup_{x\in\Omega} \rho_{\partial M}(x).$$

Then we have

$$m_f(\Omega) \leq \sup_{t\in(D_1(\Omega),D_2(\Omega))} \frac{\int_t^{D_2(\Omega)} s^{N-1}_{\kappa,\lambda}(a)\,da}{s^{N-1}_{\kappa,\lambda}(t)}\, m_{f,\partial\Omega}(\partial\Omega).$$

Proof. Define a function $\hat{\varphi} : [D_1(\Omega), D_2(\Omega)] \to \mathbb{R}$ by

$$\hat{\varphi}(t) := \int_{D_1(\Omega)}^t \frac{\int_a^{D_2(\Omega)} s^{N-1}_{\kappa,\lambda}(b)\,db}{s^{N-1}_{\kappa,\lambda}(a)}\,da.$$

Put $\Phi := \hat{\varphi} \circ \rho_{\partial M}$. By Lemma 5.1, on $\Omega \setminus \mathrm{Cut}\,\partial M$ we have

$$\Delta_f \Phi \geq -\left(\hat{\varphi}'' - H_{N,\kappa,\lambda}\,\hat{\varphi}'\right) \circ \rho_{\partial M} = 1. \tag{10.1}$$

By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of compact subsets of the closure $\bar{\Omega}$ of Ω such that for every k, the boundary $\partial\Omega_k$ of Ω_k is a smooth hypersurface in M except for a null set in $(\partial\Omega, m_{f,\partial\Omega})$, and satisfying the following: (1) for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $\bar{\Omega} \setminus \mathrm{Cut}\,\partial M = \bigcup_{k\in\mathbb{N}} \Omega_k$; (3) for every $k \in \mathbb{N}$, and for almost every point $x \in \partial\Omega_k \cap \partial\Omega$ in $(\partial\Omega, m_{f,\partial\Omega})$, there exists a unique unit outer normal vector for Ω_k at x that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at x; (4) for every $k \in \mathbb{N}$, on $\partial\Omega_k \setminus \partial\Omega$, there exists a unique unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial\Omega$, we put

$$m_{f,k} := e^{-f|_{\partial\Omega_k \setminus \partial\Omega}} \, \mathrm{vol}_k\,.$$

Let $\nu_{\partial\Omega}$ be the unit outer normal vector on $\partial\Omega$ for Ω. Integrate the both sides of (10.1) on Ω_k. By the Green formula,

$$\begin{aligned} m_f\,(\Omega_k) &\leq \int_{\Omega_k} \Delta_f\,\Phi\,d\,m_f \\ &= -\int_{\partial\Omega_k \setminus \partial\Omega} g(\nu_k, \nabla\Phi)\,d\,m_{f,k} - \int_{\partial\Omega_k \cap \partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\,d\,m_{f,\partial\Omega}. \end{aligned}$$

Since $g(\nu_k, \nabla\Phi) \geq 0$ on $\partial\Omega_k \setminus \partial\Omega$, we have

$$m_f\,(\Omega_k) \leq -\int_{\partial\Omega_k \cap \partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\,d\,m_{f,\partial\Omega}.$$

From the Cauchy-Schwarz inequality we derive

$$\begin{aligned} m_f\,(\Omega_k) &\leq \int_{\partial\Omega_k \cap \partial\Omega} \left(\hat{\varphi}' \circ \rho_{\partial M}\right) |g(\nu_{\partial\Omega}, \nabla\rho_{\partial M})|\,d\,m_{f,\partial\Omega} \\ &\leq \sup_{t \in (D_1(\Omega), D_2(\Omega))} \hat{\varphi}'(t)\,m_{f,\partial\Omega}\,(\partial\Omega). \end{aligned}$$

By letting $k \to \infty$, we complete the proof. □

Kasue [26] has obtained Proposition 10.1 in the case where $f = 0$ and $N = n$.

Under the curvature bound (1.4), we have the following:

Proposition 10.2 ([48]). *Let us assume that κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M satisfying $\partial\Omega \cap \partial M = \emptyset$. For $\delta \in \mathbb{R}$, we put*

$$D_{\delta,1}(\Omega) := \inf_{x \in \Omega} \rho_{\partial M,\delta}(x), \quad D_{\delta,2}(\Omega) := \sup_{x \in \Omega} \rho_{\partial M,\delta}(x).$$

Then we have

$$m_f(\Omega) \leq e^{2\delta} \sup_{s \in (D_{\delta,1}(\Omega), D_{\delta,2}(\Omega))} \frac{\int_s^{D_{\delta,2}(\Omega)} s^{n-1}_{\kappa,\lambda}(a)\,da}{s^{n-1}_{\kappa,\lambda}(s)}\,m_{f,\partial\Omega}\,(\partial\Omega).$$

Proof. We define a function $\hat{\varphi} : [D_{\delta,1}(\Omega), D_{\delta,2}(\Omega)] \to \mathbb{R}$ by

$$\hat{\varphi}(s) := \int_{D_{\delta,1}(\Omega)}^{s} \frac{\int_a^{D_{\delta,2}(\Omega)} s^{n-1}_{\kappa,\lambda}(b)\,db}{s^{n-1}_{\kappa,\lambda}(a)}\,da,$$

and we put $\Phi := \hat{\varphi} \circ \rho_{\partial M,\delta}$. Lemma 5.4 tells us that on $\Omega \setminus \mathrm{Cut}\,\partial M$ we see

$$\Delta_f\,\Phi \geq -e^{-4\delta}\,\left(\hat{\varphi}'' - H_{n,\kappa,\lambda}\,\hat{\varphi}'\right) \circ \rho_{\partial M,\delta} = e^{-4\delta}. \tag{10.2}$$

By Lemma 3.10, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of compact subsets of the closure $\bar{\Omega}$ of Ω such that for every k, the boundary $\partial\Omega_k$ of Ω_k is a smooth hypersurface in M except for a null set in $(\partial\Omega, m_{f,\partial\Omega})$, and satisfying the following: (1) for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\Omega_{k_1} \subset \Omega_{k_2}$; (2) $\bar{\Omega} \setminus \operatorname{Cut}\partial M = \bigcup_{k\in\mathbb{N}} \Omega_k$: (3) for every $k \in \mathbb{N}$, and for almost every point $x \in \partial\Omega_k \cap \partial\Omega$ in $(\partial\Omega, m_{f,\partial\Omega})$, there exists a unique unit outer normal vector for Ω_k at x that coincides with the unit outer normal vector on $\partial\Omega$ for Ω at x; (4) for every $k \in \mathbb{N}$, on $\partial\Omega_k \setminus \partial\Omega$, there exists a unique unit outer normal vector field ν_k for Ω_k such that $g(\nu_k, \nabla\rho_{\partial M}) \geq 0$.

For the canonical Riemannian volume measure vol_k on $\partial\Omega_k \setminus \partial\Omega$, let

$$m_{f,k} := e^{-f|_{\partial\Omega_k\setminus\partial\Omega}} \,\mathrm{vol}_k\,.$$

Let $\nu_{\partial\Omega}$ be the unit outer normal vector on $\partial\Omega$ for Ω. Integrate the both sides of (10.2) on Ω_k. Using the Green formula, we have

$$\begin{aligned} e^{-4\delta}\, m_f\,(\Omega_k) &\leq \int_{\Omega_k} \Delta_f\, \Phi\, d\,m_f \\ &= -\int_{\partial\Omega_k\setminus\partial\Omega} g(\nu_k, \nabla\Phi)\, d\,m_{f,k} - \int_{\partial\Omega_k\cap\partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\, d\,m_{f,\partial\Omega}. \end{aligned}$$

We have $g(\nu_k, \nabla\Phi) \geq 0$ on $\partial\Omega_k \setminus \partial\Omega$, and hence

$$e^{-4\delta}\, m_f\,(\Omega_k) \leq -\int_{\partial\Omega_k\cap\partial\Omega} g(\nu_{\partial\Omega}, \nabla\Phi)\, d\,m_{f,\partial\Omega}.$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned} e^{-4\delta}\, m_f\,(\Omega_k) &\leq \int_{\partial\Omega_k\cap\partial\Omega} \left(\hat{\varphi}' \circ \rho_{\partial M,\delta}\right) |g(\nu_{\partial\Omega}, \nabla\rho_{\partial M,\delta})|\, d\,m_{f,\partial\Omega} \\ &\leq e^{-2\delta} \sup_{s\in(D_{\delta,1}(\Omega), D_{\delta,2}(\Omega))} \hat{\varphi}'(s)\, m_{f,\partial\Omega}\,(\partial\Omega). \end{aligned}$$

Letting $k \to \infty$, we obtain the desired inequality. $\square$

10.2 Spectrum estimates

For $\alpha > 0$, the *f-Dirichlet α-isoperimetric constant $ID_\alpha(M, m_f)$* of M is defined as

$$ID_\alpha(M, m_f) := \inf_{\Omega} \frac{m_{f,\partial\Omega}(\partial\Omega)}{\left(m_f(\Omega)\right)^{1/\alpha}},$$

where the infimum is taken over all relatively compact domains Ω in M such that $\partial\Omega$ are smooth hypersurfaces in M satisfying $\partial\Omega \cap \partial M = \emptyset$. The *$f$-Dirichlet α-Sobolev constant $SD_\alpha(M, m_f)$* of M is defined as

$$SD_\alpha(M, m_f) := \inf_{\phi\in W_0^{1,1}(M,m_f)\setminus\{0\}} \frac{\int_M \|\nabla\phi\|\, d\,m_f}{\left(\int_M |\phi|^\alpha\, d\,m_f\right)^{1/\alpha}},$$

where the infimum is taken over all non-zero functions ϕ in $W_0^{1,1}(M, m_f)$.

Recall the following relationship between the isoperimetric constant and the Sobolev constant that has been formally established by Federer and Fleming in [15] (see e.g., [7], [31]), and later used by Cheeger in [8] for the estimate of the first Dirichlet eigenvalue of the Laplacian.

Proposition 10.3 ([15])**.** *For all $\alpha > 0$ we have*

$$ID_\alpha(M, m_f) = SD_\alpha(M, m_f).$$

A proof of Proposition 10.3 has been given in [31] in the case of $f = 0$ (see Theorem 9.5 in [31]). The method of the proof also works in our weighted setting.

For $N \in (1, \infty)$ and $D \in (0, \bar{C}_{\kappa,\lambda}]$, we put

$$C(N, \kappa, \lambda, D) := \sup_{t \in [0,D)} \frac{\int_t^D s_{\kappa,\lambda}^{N-1}(a)\, da}{s_{\kappa,\lambda}^{N-1}(t)}. \tag{10.3}$$

Note that $C(N, \kappa, \lambda, \infty)$ is finite if and only if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$; in this case,

$$C(N, \kappa, \lambda, D) = ((N-1)\lambda)^{-1} \left(1 - e^{-(N-1)\lambda D}\right).$$

From Proposition 10.1 we derive the following spectrum estimate:

Theorem 10.4 ([46])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. For $N \in [n, \infty)$, suppose $\operatorname{Ric}_{f,M}^N \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. For $D \in (0, \bar{C}_{\kappa,\lambda}]$, suppose $D(M, \partial M) \leq D$. Then*

$$\mu_{f,p}(M) \geq (\, p\, C(N, \kappa, \lambda, D)\,)^{-p},$$

where $C(N, \kappa, \lambda, D)$ is the constant defined as (10.3)*.*

Proof. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M with $\partial\Omega \cap \partial M = \emptyset$. Set $C := C(N, \kappa, \lambda, D)$. Proposition 10.1 implies $m_f(\Omega) \leq C\, m_{f,\partial\Omega}(\partial\Omega)$. By Proposition 10.3, we obtain $SD_1(M, m_f) \geq C^{-1}$. For all $\phi \in W_0^{1,1}(M, m_f)$ we have

$$\int_M |\phi|\, dm_f \leq C \int_M \|\nabla\phi\|\, dm_f. \tag{10.4}$$

Let ψ be a non-zero function in $W_0^{1,p}(M, m_f)$. Put $q := p\,(1-p)^{-1}$. In (10.4), by replacing ϕ with $|\psi|^p$, and by the Hölder inequality, we see

$$\begin{aligned} \int_M |\psi|^p\, dm_f &\leq p\, C \int_M |\psi|^{p-1}\, \|\nabla\psi\|\, dm_f \\ &\leq p\, C \left(\int_M |\psi|^p\, dm_f\right)^{1/q} \left(\int_M \|\nabla\psi\|^p\, dm_f\right)^{1/p}. \end{aligned}$$

Considering the Rayleigh quotient $R_{f,p}(\psi)$, we complete the proof. □

Under the curvature bound (1.4), we have the following:

Theorem 10.5 ([48])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Suppose that ∂M is compact. Let $p \in (1, \infty)$. Let κ and λ satisfy the monotone-condition. For $N \in (-\infty, 1]$, suppose $\operatorname{Ric}_{f,M}^N \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M. For $D \in (0, \bar{C}_{\kappa,\lambda}]$, suppose $D_\delta(M, \partial M) \leq D$. Then we have*

$$\mu_{f,p}(M) \geq (\, p\, e^{2\delta}\, C(n, \kappa, \lambda, D)\,)^{-p},$$

where $C(n, \kappa, \lambda, D)$ is the constant defined as (10.3)*.*

Proof. Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M with $\partial\Omega \cap \partial M = \emptyset$. If we put $C_\delta := e^{2\delta}\, C(n,\kappa,\lambda,D)$, then by Proposition 10.2, we have $m_f(\Omega) \leq C_\delta\, m_{f,\partial\Omega}(\partial\Omega)$. From Proposition 10.3 we derive $SD_1(M,m_f) \geq C_\delta^{-1}$, and hence for all $\phi \in W_0^{1,1}(M,m_f)$ we see

$$\int_M |\phi|\, d\,m_f \leq C_\delta \int_M \|\nabla \phi\|\, d\,m_f. \tag{10.5}$$

Let ψ be a non-zero function in $W_0^{1,p}(M,m_f)$, and let $q := p\,(1-p)^{-1}$. In (10.5), we replace ϕ with $|\psi|^p$. Using the Hölder inequality, we have

$$\begin{aligned} \int_M |\psi|^p\, d\,m_f &\leq p\, C_\delta \int_M |\psi|^{p-1}\, \|\nabla \psi\|\, d\,m_f \\ &\leq p\, C_\delta \left(\int_M |\psi|^p\, d\,m_f \right)^{1/q} \left(\int_M \|\nabla \psi\|^p\, d\,m_f \right)^{1/p}. \end{aligned}$$

By considering the Rayleigh quotient $R_{f,p}(\psi)$, we obtain the desired inequality. $\square$

10.3 Spectrum rigidity

We prove Theorem 1.11.

Proof of Theorem 1.11. Suppose that ∂M is compact. Let $p \in (1,\infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in [n,\infty)$, suppose $\mathrm{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$.

We see $C(N,\kappa,\lambda,D) = ((N-1)\lambda)^{-1} \left(1 - e^{-(N-1)\lambda D}\right)$, where $C(N,\kappa,\lambda,D)$ is defined as (10.3). Note that the right hand side is monotone increasing as $D \to \infty$. Put $\hat{D} := D(M,\partial M)$. From Theorem 10.4 we derive

$$\mu_{f,p}(M) \geq (\, p\, C(N,\kappa,\lambda,\hat{D})\,)^{-p} \geq \left(\frac{(N-1)\lambda}{p} \right)^p.$$

We obtain (1.28).

Assume that the equality in (1.28) holds. By the monotonicity of $C(N,\kappa,\lambda,D)$ with respect to D, we have $\hat{D} = \infty$. Since ∂M is compact, Lemma 3.6 tells us that M is non-compact. Due to Corollary 7.5, we conclude Theorem 1.11. This completes the proof of Theorem 1.11. $\square$

Next, we prove Theorem 1.12.

Proof of Theorem 1.12. Suppose that ∂M is compact. Let $p \in (1,\infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $N \in (-\infty,1]$, suppose that we have $\mathrm{Ric}^N_{f,M} \geq (n-1)\kappa\, e^{\frac{-4f}{n-1}}$ and $H_{f,\partial M} \geq (n-1)\lambda e^{\frac{-2f}{n-1}}$. For $\delta \in \mathbb{R}$, suppose $f \leq (n-1)\delta$ on M.

It holds that $C(n,\kappa,\lambda,D) = ((n-1)\lambda)^{-1} \left(1 - e^{-(n-1)\lambda D}\right)$. The right hand side is monotone increasing as $D \to \infty$. Set $D_\delta := D_\delta(M,\partial M)$. By Theorem 10.5,

$$\mu_{f,p}(M) \geq e^{-2p\delta}\, (\, p\, C(n,\kappa,\lambda,D_\delta)\,)^{-p} \geq e^{-2p\delta} \left(\frac{(n-1)\lambda}{p} \right)^p.$$

This proves (1.29).

Assume that the equality in (1.29) holds. From the monotonicity of $C(n,\kappa,\lambda,D)$, it follows that $D_\delta = \infty$; in particular, $D(M,\partial M) = \infty$. By Lemma 3.6, M is non-compact. Using Corollary 7.6, we complete the proof of Theorem 1.12. $\square$

Chapter 11

Segment inequalities and eigenvalue estimates

In this chapter, we show a segment inequality of Cheeger-Colding type under the assumption that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$, where Ric_M denotes the infimum of Ric_g on the unit tangent bundle of M, and let $H_{\partial M} := \inf_{z \in \partial M} H_z$. Using the segment inequality, we give a lower bound of the smallest Dirichlet eigenvalue for the p-Laplacian that is controlled by a constant defined as follows: For $N \in (1, \infty)$ and $D \in (0, \bar{C}_{\kappa,\lambda}]$,

$$C_1(N, \kappa, \lambda, D) := \sup_{a \in (0,D)} \sup_{b \in (0,a)} \frac{s_{\kappa,\lambda}^{N-1}(a)}{s_{\kappa,\lambda}^{N-1}(b)}. \tag{11.1}$$

11.1 Segment inequalities

We prove the following segment inequality of Cheeger-Colding type:

Proposition 11.1 ([45])**.** *Suppose* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and* $H_{\partial M} \geq (n-1)\lambda$*. For* $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$*, suppose* $D(M, \partial M) \leq D$*. Let* $\phi : M \to \mathbb{R}$ *be a non-negative integrable function on* M*, and define a function* $E_\phi : M \to \mathbb{R}$ *by*

$$E_\phi(x) := \inf_{z \in \partial M} \int_0^{\rho_{\partial M}(x)} \phi(\gamma_z(a))\, da, \tag{11.2}$$

where the infimum is taken over all foot points z *on* ∂M *of* x*. Then*

$$\int_M E_\phi \, d\,\mathrm{vol}_g \leq C_1(n, \kappa, \lambda, D)\, D \int_M \phi \, d\,\mathrm{vol}_g, \tag{11.3}$$

where $C_1(n, \kappa, \lambda, D)$ *is the constant defined as* (11.1)*.*

Proof. Put $C_1 := C_1(n, \kappa, \lambda, D)$. By Lemma 4.4, for all $z \in \partial M$ and $a \in (0, \tau(z))$ we see

$$E_\phi(\gamma_z(a))\, \theta(a, z) \leq C_1 \int_0^a \phi(\gamma_z(b))\, \theta(b, z)\, db. \tag{11.4}$$

Integrating the both sides of (11.4) over $(0, \tau(z))$ with respect to a, we have

$$\int_0^{\tau(z)} E_\phi(\gamma_z(a))\, \theta(a, z)\, da \leq C_1 D \int_0^{\tau(z)} \phi(\gamma_z(b))\, \theta(b, z)\, db. \tag{11.5}$$

We integrate the both sides of (11.5) over ∂M with respect to z. By Lemma 3.5 and the Fubini theorem, we arrive at the desired inequality (11.3). $\square$

11.2 Poincaré inequalities

From Proposition 11.1 we derive the following Poincaré inequality:

Lemma 11.2 ([45]). *Suppose that we have* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and* $H_{\partial M} \geq (n-1)\lambda$. *For* $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, *suppose* $D(M, \partial M) \leq D$. *Let* $\psi : M \to \mathbb{R}$ *be a smooth integrable function on* M *with* $\psi|_{\partial M} = 0$. *Assume* $\int_M \|\nabla\psi\| \, d\,\mathrm{vol}_g < \infty$. *Then*

$$\int_M |\psi| \, d\,\mathrm{vol}_g \leq C_1(n,\kappa,\lambda,D) D \int_M \|\nabla\psi\| \, d\,\mathrm{vol}_g, \tag{11.6}$$

where $C_1(n,\kappa,\lambda,D)$ *is defined as* (11.1).

Proof. Put $\phi := \|\nabla\psi\|$, and let E_ϕ be the function defined as (11.2). Using the Cauchy-Schwarz inequality and $\psi|_{\partial M} = 0$, for each $x \in D_{\partial M}$ we see

$$|\psi(x)| = |\psi(x) - \psi(z)| \leq \int_0^{\rho_{\partial M}(x)} \left| g(\nabla\psi, \gamma_z'(t)) \right| \, dt \leq E_\phi(x),$$

where z denotes a unique foot point on ∂M of x. We integrate the both sides over $D_{\partial M}$ with respect to x. By Propositions 3.9 and 11.1, we obtain (11.6). $\square$

11.3 Eigenvalue estimates

As one of the applications of our segment inequality, we have the following:

Proposition 11.3 ([45]). *Suppose that* M *is compact. Let* $p \in (1,\infty)$. *Suppose* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and* $H_{\partial M} \geq (n-1)\lambda$. *For* $D \in (0, \bar{C}_{\kappa,\lambda}]$, *suppose* $D(M, \partial M) \leq D$. *Then we have*

$$\mu_{0,p}(M) \geq (p\, C_1(n,\kappa,\lambda,D)\, D)^{-p}, \tag{11.7}$$

where $C_1(n,\kappa,\lambda,D)$ *is defined as* (11.1).

Proof. Let ψ be a non-zero function in $W_0^{1,p}(M, \mathrm{vol}_g)$. We may assume that ψ is smooth on M. Put $q := p\,(1-p)^{-1}$. In Lemma 11.2, by replacing ψ with $|\psi|^p$, and by the Hölder inequality, we see

$$\begin{aligned} &\int_M |\psi|^p \, d\,\mathrm{vol}_g \leq p\, C_1(n,\kappa,\lambda,D)\, D \int_M |\psi|^{p-1} \|\nabla\psi\| \, d\,\mathrm{vol}_g \\ &\leq p\, C_1(n,\kappa,\lambda,D)\, D \left(\int_M |\psi|^p \, d\,\mathrm{vol}_g \right)^{1/q} \left(\int_M \|\nabla\psi\|^p \, d\,\mathrm{vol}_g \right)^{1/p}. \end{aligned}$$

Considering the Rayleigh quotient, we conclude (11.7). $\square$

Remark 11.1. Proposition 11.3 is weaker than Theorem 10.4 in the case where $f = 0$ and $N = n$. We can prove that the lower bound $(p\, C_1(n,\kappa,\lambda,D)\, D)^{-p}$ for $\mu_{0,p}$ in Proposition 11.3 is at most the lower bound $(p\, C(n,\kappa,\lambda,D))^{-p}$ in Theorem 10.4.

Chapter 12

Measure contraction properties

In this chapter, we study measure contraction properties of manifolds with boundary. We prove a measure contraction inequality around the boundary under the assumption that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$, where Ric_M is the infimum of Ric_g on the unit tangent bundle of M, and $H_{\partial M} := \inf_{z \in \partial M} H_z$. By using the measure contraction inequality, we give another proof of Theorem 8.15 in the case where $f = 0$ and $N = n$.

12.1 Measure contraction inequalities

Let $t \in (0,1)$. For a point $x \in M$, we say that $y \in M$ is a *t-extension point from ∂M of x* if y satisfies the following: (1) $\rho_{\partial M}(x)/\rho_{\partial M}(y) = t$; (2) there exists a foot point z on ∂M of x with $y = \gamma_z(\rho_{\partial M}(y))$. Let W_t denote the set of all points $x \in M$ of which there exists a t-extension point from ∂M.

We have the following property of t-extension points:

Lemma 12.1 ([45])**.** *Let $t \in (0,1)$. For every point $x \in W_t$, there exists a unique foot point on ∂M of x. In particular, every point $x \in W_t$ has a unique t-extension point from ∂M.*

Proof. Let $x \in W_t$, and let y be a t-extension point from ∂M of x. We have a foot point z on ∂M of x such that $y = \gamma_z(\rho_{\partial M}(y))$. By $\rho_{\partial M}(y) \leq \tau(z)$ and $\rho_{\partial M}(x) = t\rho_{\partial M}(y)$, we have $\rho_{\partial M}(x) < \tau(z)$. Furthermore, Lemma 3.1 implies $x = \gamma_z(\rho_{\partial M}(x))$. By Lemma 3.3, z is a unique foot point on ∂M of x.

Suppose that for x, we have distinct t-extension points $y_1, y_2 \in M$ from ∂M. Then we have $\rho_{\partial M}(y_1) = \rho_{\partial M}(y_2)$. For each $i = 1, 2$, we have a foot point z_i on ∂M of x such that $y_i = \gamma_{z_i}(\rho_{\partial M}(y_i))$. From $y_1 \neq y_2$ we deduce $z_1 \neq z_2$. This is a contradiction since x has a unique foot point on ∂M. $\square$

Lemma 12.1 tells us that for $t \in (0,1)$, we can define a map $\Phi_t : W_t \to M$ by $\Phi_t(x) := y$, where y is a unique t-extension point from ∂M of x. We call Φ_t the *t-extension map from ∂M*. We see that the t-extension map Φ_t from ∂M is surjective and continuous.

For $\Omega \subset M$, we recall that $z \in \partial M$ is said to be a *foot point on ∂M of* Ω if there exists $x \in \Omega$ such that z is a foot point on ∂M of x. We denote by $\Pi(\Omega)$ the set of all foot points on ∂M of Ω.

We prove the following property of the t-extension map from ∂M:

Lemma 12.2 ([45]). *For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. Let Ω be a subset of M. Then we have $\Pi(\Phi_t^{-1}(\Omega)) = \Pi(\Omega)$.*

Proof. We show $\Pi(\Omega) \subset \Pi(\Phi_t^{-1}(\Omega))$. Let $z \in \Pi(\Omega)$. We have $x \in \Omega$ such that z is a foot point on ∂M of x. We put $x_t := \gamma_z(t\rho_{\partial M}(x))$. It suffices to show that z is a foot point on ∂M of x_t, and $x_t \in \Phi_t^{-1}(\Omega)$. From Lemma 3.1 we deduce $x = \gamma_z(\rho_{\partial M}(x))$. Since $\rho_{\partial M}(x) \leq \tau(z)$, we have $t\rho_{\partial M}(x) < \tau(z)$. Lemma 3.3 implies that z is a unique foot point on ∂M of x_t. Furthermore, we see $\rho_{\partial M}(x_t) = t\rho_{\partial M}(x)$. It follows that x is a t-extension point from ∂M of x_t. By $x = \Phi_t(x_t)$ and $x \in \Omega$, we obtain $x_t \in \Phi_t^{-1}(\Omega)$. This proves $z \in \Pi(\Phi_t^{-1}(\Omega))$.

We next show the opposite. Let $z \in \Pi(\Phi_t^{-1}(\Omega))$. We have a point $x \in \Phi_t^{-1}(\Omega)$ such that z is a foot point on ∂M of x. Lemma 12.1 tells us that z is a unique foot point on ∂M of x. We see $\Phi_t(x) = \gamma_z(\rho_{\partial M}(\Phi_t(x)))$, and hence $\rho_{\partial M}(\Phi_t(x)) \leq \tau(z)$. We conclude that z is a foot point on ∂M of $\Phi_t(x)$. From $\Phi_t(x) \in \Omega$ we derive $z \in \Pi(\Omega)$. This completes the proof. □

For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. For a subset Ω of M, and for $z \in \Pi(\Omega)$, put

$$I_{\Omega,t,z} := \{\, a \in (0, t\tau(z)) \mid \gamma_z(a) \in \Phi_t^{-1}(\Omega) \,\}.$$

We have the following integration formula:

Lemma 12.3 ([45]). *For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. Suppose that a subset Ω of M is measurable, and satisfies $\operatorname{vol}_g \Phi_t^{-1}(\Omega) < \infty$. Then we have*

$$\operatorname{vol}_g \Phi_t^{-1}(\Omega) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,z}} \theta(a,z)\, da\, d\operatorname{vol}_h .$$

Proof. We put

$$A := \{\, \gamma_z(t\tau(z)) \in \Phi_t^{-1}(\Omega) \mid z \in \Pi(\Omega), \tau(z) < \infty \},$$
$$B := \{\, \gamma_z(a) \mid z \in \Pi(\Omega), a \in I_{\Omega,t,z} \}.$$

Note that A and B are disjoint.

We prove $\Phi_t^{-1}(\Omega) \setminus \partial M = A \sqcup B$. The inclusion $A \sqcup B \subset \Phi_t^{-1}(\Omega) \setminus \partial M$ follows from the definition of $I_{\Omega,t,z}$. We show the opposite. Take $x \in \Phi_t^{-1}(\Omega) \setminus \partial M$, and a foot point z on ∂M of x. Lemma 3.1 implies $x = \gamma_z(\rho_{\partial M}(x))$. From Lemma 12.2 we deduce $z \in \Pi(\Omega)$. By $x \in W_t$, and by Lemma 12.1, z is a unique foot point on ∂M of x, and we have a unique t-extension point $y \in M$ from ∂M of x. We see $t\rho_{\partial M}(y) = \rho_{\partial M}(x)$ and $y = \gamma_z(\rho_{\partial M}(y))$. From $\rho_{\partial M}(y) \leq \tau(z)$ we derive $\rho_{\partial M}(x) \leq t\tau(z)$. We have the opposite inclusion.

We next prove that A is a null set of M. We put

$$\widehat{A} := \bigcup_{z \in \Pi(\Omega)} \{t\tau(z)u_z \mid \tau(z) < \infty\}.$$

Note that $A = \exp^{\perp}(\widehat{A})$. By Lemma 3.4, and by the Fubini theorem, the graph $\{(z, t\tau(z)) \mid z \in \partial M, \tau(z) < \infty\}$ of $t\tau$ is a null set of $\partial M \times [0,\infty)$. Since a map $\Phi : \partial M \times [0,\infty) \to T^{\perp}\partial M$ defined by $\Phi(z,a) := au_z$ is smooth, the set $\widehat{A}$ is also a null set of $T^{\perp}\partial M$. By the smoothness of $\exp^{\perp}$, the set A is a null set.

By $\Phi_t^{-1}(\Omega) \setminus \partial M = A \sqcup B$ and $\mathrm{vol}_g A = 0$, it suffices to show that

$$\mathrm{vol}_g B = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,z}} \theta(a,z)\, da\, d\,\mathrm{vol}_h .$$

Put

$$\widehat{B} := \bigcup_{z \in \Pi(\Omega)} \{ a u_z \mid a \in I_{\Omega,t,z} \}.$$

Notice that $\widehat{B}$ is contained in $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$, and that $B = \exp^{\perp}(\widehat{B})$. By Lemma 3.7, and by using the coarea formula and the Fubini theorem, we have

$$\mathrm{vol}_g B = \mathrm{vol}_g \exp^{\perp}(\widehat{B}) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,z}} \theta(a,z)\, da\, d\,\mathrm{vol}_h .$$

Therefore, we obtain the desired formula. □

We now prove the following measure contraction inequality:

Proposition 12.4 ([45]). *Suppose* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and* $H_{\partial M} \geq (n-1)\lambda$. *For* $t \in (0,1)$, *let* Φ_t *be the* t*-extension map from* ∂M. *Suppose that a subset* Ω *of* M *is measurable. Then we have*

$$\mathrm{vol}_g \Phi_t^{-1}(\Omega) \geq t \int_{\Omega} \frac{s_{\kappa,\lambda}^{n-1} \circ t\rho_{\partial M}}{s_{\kappa,\lambda}^{n-1} \circ \rho_{\partial M}}\, d\,\mathrm{vol}_g .$$

Proof. We may assume that $\mathrm{vol}_g \Phi_t^{-1}(\Omega)$ is finite. By Lemma 12.3, we see

$$\mathrm{vol}_g \Phi_t^{-1}(\Omega) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,z}} \theta(a,z)\, da\, d\,\mathrm{vol}_h . \tag{12.1}$$

From Lemma 8.1, for all $z \in \Pi(\Omega)$ and $a \in I_{\Omega,t,z}$ we deduce

$$\frac{\theta(t^{-1}a, z)}{\theta(a,z)} \leq \frac{s_{\kappa,\lambda}^{n-1}(t^{-1}a)}{s_{\kappa,\lambda}^{n-1}(a)}. \tag{12.2}$$

Using (12.1) and (12.2), we have

$$\mathrm{vol}_g \Phi_t^{-1}(\Omega) \geq \int_{\Pi(\Omega)} \int_{I_{\Omega,t,z}} \frac{s_{\kappa,\lambda}^{n-1}(a)}{s_{\kappa,\lambda}^{n-1}(t^{-1}a)} \theta(t^{-1}a, z)\, da\, d\,\mathrm{vol}_h . \tag{12.3}$$

For $z \in \Pi(\Omega)$, we put

$$I_{\Omega,z} := \{ a \in (0, \tau(z)) \mid \gamma_z(a) \in \Omega \}$$

that coincides with the set $\{ b \in (0,\tau(z)) \mid tb \in I_{\Omega,t,z} \}$. Putting $b := t^{-1}a$ in (12.3), we derive

$$\mathrm{vol}_g \Phi_t^{-1}(\Omega) \geq t \int_{\Pi(\Omega)} \int_{I_{\Omega,z}} \frac{s_{\kappa,\lambda}^{n-1}(tb)}{s_{\kappa,\lambda}^{n-1}(b)} \theta(b,z)\, db\, d\,\mathrm{vol}_h . \tag{12.4}$$

For the set

$$U := \bigcup_{z \in \Pi(\Omega)} \{ a u_z \mid a \in I_{\Omega,z} \},$$

we show $\exp^{\perp}(U) = \Omega \setminus (\mathrm{Cut}\, \partial M \cup \partial M)$. We see that the inclusion

$$\exp^{\perp}(U) \subset \Omega \setminus (\mathrm{Cut}\, \partial M \cup \partial M)$$

follows from the definition of $I_{\Omega,z}$. We show the opposite inclusion. Take a point $x \in \Omega \setminus (\mathrm{Cut}\, \partial M \cup \partial M)$, and a foot point z on ∂M of x. Lemma 3.1 implies $x = \gamma_z(\rho_{\partial M}(x))$. From $x \notin \mathrm{Cut}\, \partial M \cup \partial M$ we deduce $\rho_{\partial M}(x) \in (0, \tau(z))$. Since $z \in \Pi(\Omega)$, we have $\rho_{\partial M}(x) \in I_{\Omega,z}$. This proves the opposite

The set U is contained in $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$. By Lemma 3.7 and Proposition 3.9, and by using the coarea formula and the Fubini theorem, we obtain

$$\begin{aligned} t \int_{\Pi(\Omega)} \int_{I_{\Omega,z}} \frac{s^{n-1}_{\kappa,\lambda}(tb)}{s^{n-1}_{\kappa,\lambda}(b)} \theta(b,z)\, db\, d\,\mathrm{vol}_h &= t \int_{\exp^{\perp}(U)} \frac{s^{n-1}_{\kappa,\lambda} \circ t\rho_{\partial M}}{s^{n-1}_{\kappa,\lambda} \circ \rho_{\partial M}} d\,\mathrm{vol}_g \\ &= t \int_{\Omega} \frac{s^{n-1}_{\kappa,\lambda} \circ t\rho_{\partial M}}{s^{n-1}_{\kappa,\lambda} \circ \rho_{\partial M}} d\,\mathrm{vol}_g . \end{aligned} \tag{12.5}$$

Combining (12.4) and (12.5), we complete the proof. □

12.2 Another proof of the relative volume comparison

For $r, R > 0$ with $r < R$, we put $A_{r,R}(\partial M) := B_R(\partial M) \setminus B_r(\partial M)$.

From Proposition 12.4 we derive the following:

Lemma 12.5 ([45])**.** *Let ∂M be compact. Let $t \in (0,1)$. Suppose $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$. Then for all $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ and $r \in (0, R)$,*

$$\frac{\mathrm{vol}_g A_{r,R}(\partial M)}{\mathrm{vol}_g A_{tr,tR}(\partial M)} \leq \left(t \inf_{a \in (r,R)} \frac{s^{n-1}_{\kappa,\lambda}(ta)}{s^{n-1}_{\kappa,\lambda}(a)} \right)^{-1}.$$

Proof. Let Φ_t be the t-extension map from ∂M. Put $\Omega := A_{r,R}(\partial M)$. For every $x \in \Phi_t^{-1}(\Omega)$, we have

$$\rho_{\partial M}(x) = t\, \rho_{\partial M}(\Phi_t(x)) \in (tr, tR];$$

in particular, $\Phi_t^{-1}(\Omega) \subset A_{tr,tR}(\partial M)$. By applying Proposition 12.4 to Ω, we see

$$\mathrm{vol}_g A_{tr,tR}(\partial M) \geq \mathrm{vol}_g \Phi_t^{-1}(\Omega) \geq t \inf_{a \in (r,R)} \frac{s^{n-1}_{\kappa,\lambda}(ta)}{s^{n-1}_{\kappa,\lambda}(a)}\, \mathrm{vol}_g \Omega.$$

We obtain the desired inequality. □

Lemma 12.5 implies the following:

Lemma 12.6 ([45])**.** *Suppose that ∂M is compact. Let $r_2 \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, and let $r_1 \in (0, r_2)$. Put $t := r_1/r_2$. For $k \in \mathbb{N}$, put $r := t^k r_2$. Suppose $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$. Then we have*

$$\frac{\mathrm{vol}_g A_{r_1,r_2}(\partial M)}{\mathrm{vol}_g B_r(\partial M)} \leq \left(\sum_{i=k}^{\infty} t^i \inf_{a \in (r_1,r_2)} \frac{s^{n-1}_{\kappa,\lambda}(t^i a)}{s^{n-1}_{\kappa,\lambda}(a)} \right)^{-1}.$$

Proof. It holds that $B_r(\partial M) \setminus \partial M = \bigcup_{i=k}^{\infty} A_{t^i r_1, t^i r_2}(\partial M)$. By Lemma 12.5, we see

$$\begin{aligned}\operatorname{vol}_g B_r(\partial M) &= \sum_{i=k}^{\infty} \operatorname{vol}_g A_{t^i r_1, t^i r_2}(\partial M) \\ &\geq \operatorname{vol}_g A_{r_1, r_2}(\partial M) \left(\sum_{i=k}^{\infty} t^i \inf_{a \in (r_1, r_2)} \frac{s_{\kappa,\lambda}^{n-1}(t^i a)}{s_{\kappa,\lambda}^{n-1}(a)} \right).\end{aligned}$$

This proves the lemma. $\square$

Furthermore, we have the following:

Lemma 12.7. *Suppose that ∂M is compact. Let $t \in (0,1)$. Take $l, m \in \mathbb{N}$ with $l < m$. Suppose $\operatorname{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$. Then for all $r > 0$ with $t^{l-1} r \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, we have*

$$\frac{\operatorname{vol}_g B_{t^{l-1}r}(\partial M)}{\operatorname{vol}_g B_{t^{m-1}r}(\partial M)} \leq \frac{\sum_{j=l}^{\infty} \sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)(t^{j-1} r - t^j r)}{\sum_{i=m}^{\infty} \inf_{a \in (t^i r, t^{i-1} r)} s_{\kappa,\lambda}^{n-1}(a)(t^{i-1} r - t^i r)}.$$

Proof. We fix $j \in \{l, \ldots, m-1\}$. Lemma 12.6 tells us that

$$\begin{aligned}\frac{\operatorname{vol}_g A_{t^j r, t^{j-1} r}(\partial M)}{\operatorname{vol}_g B_{t^{m-1} r}(\partial M)} &\leq \left(\sum_{i=m-j}^{\infty} t^i \inf_{a \in (t^j r, t^{j-1} r)} \frac{s_{\kappa,\lambda}^{n-1}(t^i a)}{s_{\kappa,\lambda}^{n-1}(a)} \right)^{-1} \\ &\leq \left(\sum_{i=m-j}^{\infty} t^i \frac{\inf_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(t^i a)}{\sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)} \right)^{-1}.\end{aligned}$$

We notice that

$$\left(\sum_{i=m-j}^{\infty} t^i \frac{\inf_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(t^i a)}{\sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)} \right)^{-1} = \frac{t^j \sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)}{\sum_{i=m}^{\infty} t^i \inf_{a \in (t^i r, t^{i-1} r)} s_{\kappa,\lambda}^{n-1}(a)}.$$

Hence we have

$$\begin{aligned}\frac{\operatorname{vol}_g B_{t^{l-1} r}(\partial M)}{\operatorname{vol}_g B_{t^{m-1} r}(\partial M)} &= 1 + \sum_{j=l}^{m-1} \frac{\operatorname{vol}_g A_{t^j r, t^{j-1} r}(\partial M)}{\operatorname{vol}_g B_{t^{m-1} r}(\partial M)} \\ &\leq 1 + \sum_{j=l}^{m-1} \frac{t^j \sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)}{\sum_{i=m}^{\infty} t^i \inf_{a \in (t^i r, t^{i-1} r)} s_{\kappa,\lambda}^{n-1}(a)} \\ &\leq \frac{\sum_{j=l}^{\infty} t^j \sup_{a \in (t^j r, t^{j-1} r)} s_{\kappa,\lambda}^{n-1}(a)}{\sum_{i=m}^{\infty} t^i \inf_{a \in (t^i r, t^{i-1} r)} s_{\kappa,\lambda}^{n-1}(a)}.\end{aligned}$$

We complete the proof. $\square$

We give another proof of Theorem 8.15 in the case where $f = 0$ and $N = n$.

Proof of Theorem 8.15. Suppose that ∂M is compact. Suppose $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq (n-1)\lambda$. Let $r, R > 0$ satisfy $r \leq R$. By Proposition 6.1, we may assume that $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ and $r < R$. Put $r_0 := Rr$. Take a sufficiently large $L \in \mathbb{N}$ satisfying $L^{-1} \log r \in (0,1)$. Put

$$t := 1 - \frac{\log r}{L}, \quad l := L + 1, \quad m := \min \left\{ i \in \mathbb{N} \;\middle|\; i \geq L \frac{\log R}{\log r} + 1 \right\}.$$

We see $l < m$ and $t^{m-1} r_0 \leq r$. Notice that if $L \to \infty$, then $t^{l-1} r_0 \to R$ and $t^{m-1} r_0 \to r$. By Lemma 12.7, we have

$$\begin{aligned}
\frac{\mathrm{vol}_g B_{t^{l-1} r_0}(\partial M)}{\mathrm{vol}_g B_r(\partial M)} &\leq \frac{\mathrm{vol}_g B_{t^{l-1} r_0}(\partial M)}{\mathrm{vol}_g B_{t^{m-1} r_0}(\partial M)} \\
&\leq \frac{\sum_{j=l}^{\infty} \sup_{a \in (t^j r_0, t^{j-1} r_0)} s_{\kappa,\lambda}^{n-1}(a)(t^{j-1} r_0 - t^j r_0)}{\sum_{i=m}^{\infty} \inf_{a \in (t^i r_0, t^{i-1} r_0)} s_{\kappa,\lambda}^{n-1}(a)(t^{i-1} r_0 - t^i r_0)}.
\end{aligned}$$

By letting $L \to \infty$, we obtain

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} \leq \frac{\int_0^R s_{\kappa,\lambda}^{n-1}(a)\, da}{\int_0^r s_{\kappa,\lambda}^{n-1}(a)\, da} = \frac{s_{n,\kappa,\lambda}(R)}{s_{n,\kappa,\lambda}(r)}.$$

Thus, we have completed the proof of Theorem 8.15 when $f = 0$ and $N = n$. $\square$

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